Additional Problems for Final Exam Review

About This Review Set

As stated in the syllabus, a goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, solutions are not are provided for the additional review problem sets. This is intentional: upper level courses often don't have recitations, let alone worksheets and worksheet solutions. So students need, develop, and use various strategies to check their solutions in those courses. In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses.

The MML Study Plan

If you would like to prepare for the midterm by solving problems that have solutions, the MML Study Plan has hundreds of problems you can solve. MML will tell you if your work is correct and offers a few different study aids. To access problems for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your Math 1553 course
3. select Lay Linear Algebra (the online textbook)
4. select a chapter
5. select a section
6. click study plan

Questions on Chapter 6

1. True or False.
   (a) If y is in the subspace W and its orthogonal complement $W^\perp$, then $\vec{y}$ must be the zero vector.
   (b) If a linear system has infinitely many solutions, then it also has infinitely many least squares solutions.
   (c) The Least Squares solution $\hat{x}$ is any solution that makes $|A\hat{x} - \vec{b}|$ as small as possible.
   (d) If A and B are orthogonal square matrices, then so is $A + B$.
   (e) The null space of an orthogonal matrix is always equal to $\{0\}$.
   (f) If S is a subspace, then $\text{proj}_S\vec{u}$ is a vector in S.
   (g) For any matrix A, A^TA is the matrix of dot products of the columns of A.
   (h) If V is a subspace spanned by $\vec{v}_1$ and $\vec{v}_2$, and $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = 0$, then $\vec{x} \in V^\perp$.
   (i) If the columns of a 5×2 matrix U are orthonormal, then $UU^T\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of U.

2. W is the subspace spanned by

$$\vec{u} = \begin{pmatrix} 2 \\ 1 \\ -6 \\ 18 \end{pmatrix}$$

   Construct a basis for the orthogonal compliment $W^\perp$. What is the dimension of $W^\perp$?

3. Consider the matrix A.

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

If possible, construct a basis for the following subspaces. Compute the dimension of each space.

(a) $\text{Row}(A)$

(b) $(\text{Row}(A))^{\perp}$

(c) $\text{Col}(A)$

(d) $(\text{Col}(A))^{\perp}$

4. Fill in the blanks.

(a) If $\vec{b}$ is in column space of A, then $A\widehat{x} =$ _________.

(b) If the columns of A are linearly independent, then the square matrix $A^T A$ is __________________.

(c) The least squares solution is unique if the _________ of A are linearly independent.

(d) The Normal Equations for the Least Squares Solution are _________.

(e) If $A = Q$, then the solution $\widehat{x} =$ _________.

(f) If W is a subspace of $\mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^n$ has orthogonal projection $\widehat{y}$, then the distance from $\vec{y}$ to W is _________

(g) If the columns of A are linearly independent, then the square matrix $A^T A$ is __________________.

(h) If U is a square orthogonal matrix, then the inverse of U is $U^{-1} =$ ____.

5. Consider the data in the table below.

x	-1	0	2
y	2	0	-1

(a) Construct the normal equations that can be solved to find the values of β_1 and β_0 so that the equation of the least-squares line $y = \beta_1 x + \beta_0$ best fits the data in the table.

(b) Compute the values of β_0 and β_1.

6. Let $\vec{v}, \vec{w} \in \mathbb{R}^7$ be orthogonal vectors with $\|v\| = 5$ and $\|w\| = \sqrt{2}$. Find

$$\|\vec{x}\| = \boxed{}, \qquad \vec{x} \cdot \vec{y} = \boxed{}$$

where $\vec{x} = -\vec{v} + 3\vec{w}$ and $\vec{y} = 3\vec{v} - \vec{w}$.

7. Find an orthonormal basis for the the column space of

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

8. Complete the matrix so that all columns are orthogonal.

$$\begin{bmatrix} -1 & 1 & -1 & b \\ 1 & -1 & -1 & c \\ 0 & 1 & a & d \\ 0 & 0 & 1 & d \end{bmatrix}$$

9. Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & 5 \\ 1 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$$

   (a) Find the least squares solutions $\hat{x}$ to $A\vec{x} = \vec{b}$.

10. Find an orthogonal basis for Span$\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -1 \end{bmatrix}.$$

11. Suppose $||\mathbf{u}|| = 2$, $||\mathbf{v}|| = 3$ and $\mathbf{u} \cdot \mathbf{v} = -1$. Find the distance between $\mathbf{u}$ and $\mathbf{v}$.

12. (a) Find a and b that make $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & a & 0 \\ \frac{1}{\sqrt{3}} & b & \frac{1}{\sqrt{2}} \end{bmatrix}$ an orthogonal matrix.

   (b) Consider the following types of linear transformations of $\mathbb{R}^2$. Select all whose standard matrices are orthogonal matrices. Explain why your choices are orthogonal.

   - identity
   - shear
   - rotation
   - projection
   - reflection

13. Find the distance of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ from the plane spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. (Be careful: $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal.)

14. (a) Find the least square solutions to $Ax = b$ for the following A and b.

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

   (b) How many least square solutions are there to $Ax = b$ when A is not one-to-one?

   (c) How many least square solutions are there to $Ax = b$ when $A^T A$ is invertible?

Questions from the Center for Academic Success Reading Day

If you attended the Center for Academic Success reading day session (Wednesday Apr 26, 1:00 to 3:00 in Clough 144 and Clough 152), you may have seen these problems.

(A) The following are instructions for a linear algebra game, MatrixToe. It is similar to tic-tac-toe, and is meant for two players.

- There are two players, the 1-player (1P) and the 0-player (0P).
- 0P and 1P take turns placing numbers into an empty $N \times N$ matrix
- the game ends when all the matrix elements have a number.
- 1P: can only place 1's in the matrix, wins if the matrix is invertible.
- 0P: can only place 0's in the matrix, wins if the matrix is singular.

(a) Let $N = 3$. Decide who is the 1P and the 0P, who goes first, play a few games of Matrix Toe, and determine who won for each game.

a) Game 1: $\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$

Who won? ______ Why?

b) Game 2: $\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$

Who won? ______ Why?

c) Game 3: $\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$

Who won? ______ Why?

(b) Describe at least three strategies that the 0P might use to win.

(c) If possible, fill in the missing elements of the matrices below with numbers 0 or 1, so that each of the matrices are singular. If it is not possible to do so, state why.

$$A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

Hint: you can solve this problem by inspection. You don't need to use row reduction. A few games of MatrixToe may help you see why.

(B) If possible, give an example of the following.

(a) A matrix whose columns form an orthogonal basis for $\mathbb{R}^4$.

(b) A matrix A that is in echelon form, and

$$\dim\left((\text{Row}(A))^{\perp}\right) = 2$$
$$\dim\left((\text{Col}(A))^{\perp}\right) = 3$$

(c) A vector $\vec{v} \in \mathbb{R}^3$ and a subspace W such that $\text{proj}_W \vec{v} = \vec{v}$, and $\dim(W) = 2$.

(d) An orthogonal matrix, in echelon form, whose columns span a 2-dimensional subspace of $\mathbb{R}^3$.

(e) A matrix C such that the linear system $C\vec{x} = \vec{b}$ is inconsistent but has a unique least-squares solution, where $\vec{x} \in \mathbb{R}^3$ and

$$\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(f) A subspace S, of $\mathbb{R}^4$, that satisfies $\dim(S) = \dim(S^{\perp}) = 2$.

(g) Two linearly independent vectors that are orthogonal to $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

(h) A subspace, S, of $\mathbb{R}^3$ such that $\dim(S^{\perp}) = 2$.

(i) A 2×3 matrix whose columns are linearly independent.

(j) A 2×2 matrix that is invertible and does not have an LU decomposition.

(k) A 2×2 matrix whose eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$, and whose corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(l) An invertible 2×2 matrix whose determinant is zero.

(m) A 2×2 matrix that is diagonalizable but not invertible.

(n) A 4×3 matrix in reduced echelon form, whose columns span $\mathbb{R}^4$.

(o) A 3×3 matrix C, that is in reduced echelon form, has exactly two pivots, and satisfies

$$C \begin{pmatrix} 2 \\ -8 \\ 1 \end{pmatrix} = \vec{0}$$

.

(C) Match the items in the column on the left with the items in the column on the right. Some items match to multiple items.

(a) $\text{Proj}_{\vec{x}}\vec{y}$

(b) A set of vectors includes the zero vector.

(c) $\det A \det B$

(d) Every column of A has a pivot

(e) A basis for $\text{Col}(A)$.

(f) U is an orthogonal matrix.

(g) Orthogonal complement $W^{\perp}$

(h) $(\text{Row}A)^{\perp}$

(i) $(\text{Col}A)^{\perp}$

(j) Orthonormal vectors

(k) A is singular

(l) 0 is not an eigenvalue of A

(m) PD^kP^{-1}

(n) A is a 3×4 matrix with linearly independent columns.

(o) Orthogonal projection of $\vec{y}$ onto V

(p) A does not have an LU decomposition

(q) A has the decomposition $A = PDP^{-1}$

(r) T is a linear transformation whose standard matrix, A, is one-to-one.

(I) $\{\vec{x} \ : \ \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$

(II) $\dfrac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}\vec{x}$

(III) Unit length, pairwise orthogonal

(IV) $\det(A) = 0$

(V) Not possible

(VI) $\det(A) \neq 0$

(VII) $\left(PDP^{-1}\right)^k$

(VIII) Row swaps are needed to express A in echelon form.

(IX) $\text{Null}A$

(X) The vector $\widehat{y} \in V$ closest to $\vec{y}$.

(XI) $\text{Null}A^T$

(XII) The eigenvalues of A are distinct.

(XIII) Its columns are orthonormal.

(XIV) The vectors are linearly dependent.

(XV) The system $A\vec{x} = \vec{0}$ has only the trivial solution.

(XVI) The columns of A are linearly independent.

(XVII) The pivot columns of A.

(XVIII) $\det(AB)$

Questions on Chapters 1, 2, 3, 5