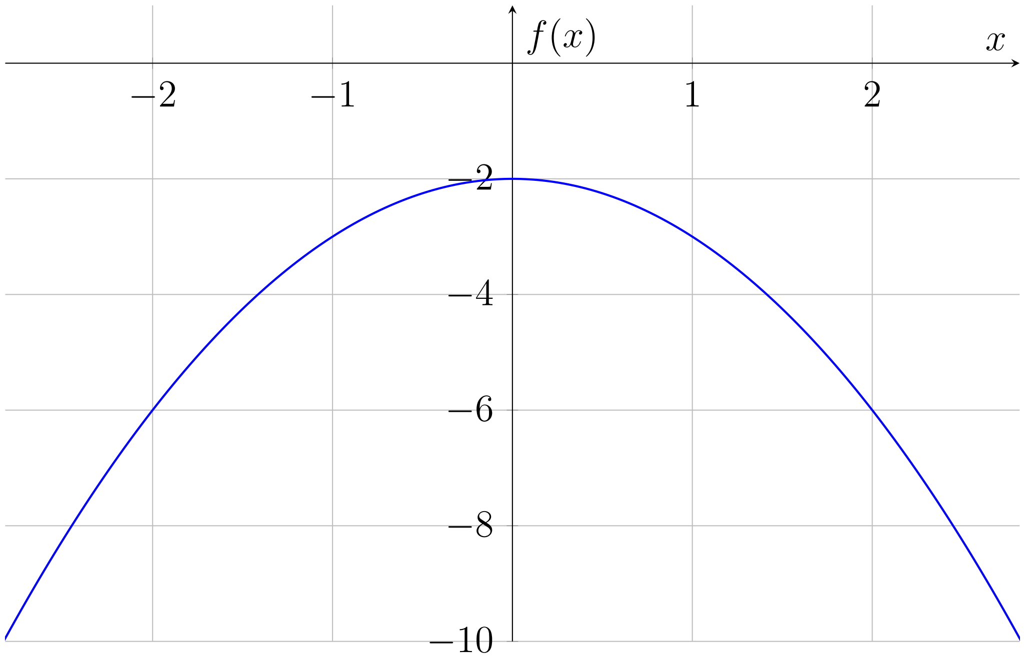
Problem 1

Graph the following function and state if it is one-to-one or not (explain how you determined this): $f(x) = -x^2 - 2$

Step 1: Graph the Function



Step 2: Determine if the Function is One-to-One

A function is one-to-one if and only if every horizontal line intersects the graph of the function at most once. This is known as the Horizontal Line Test.

Horizontal Line Test

For the function $f(x) = -x^2 - 2$, we can see that it is a downward-opening parabola. Any horizontal line $y = c$ where c is a constant will intersect the graph of $f(x)$ at most twice. Therefore, the function $f(x)$ does not pass the Horizontal Line Test.

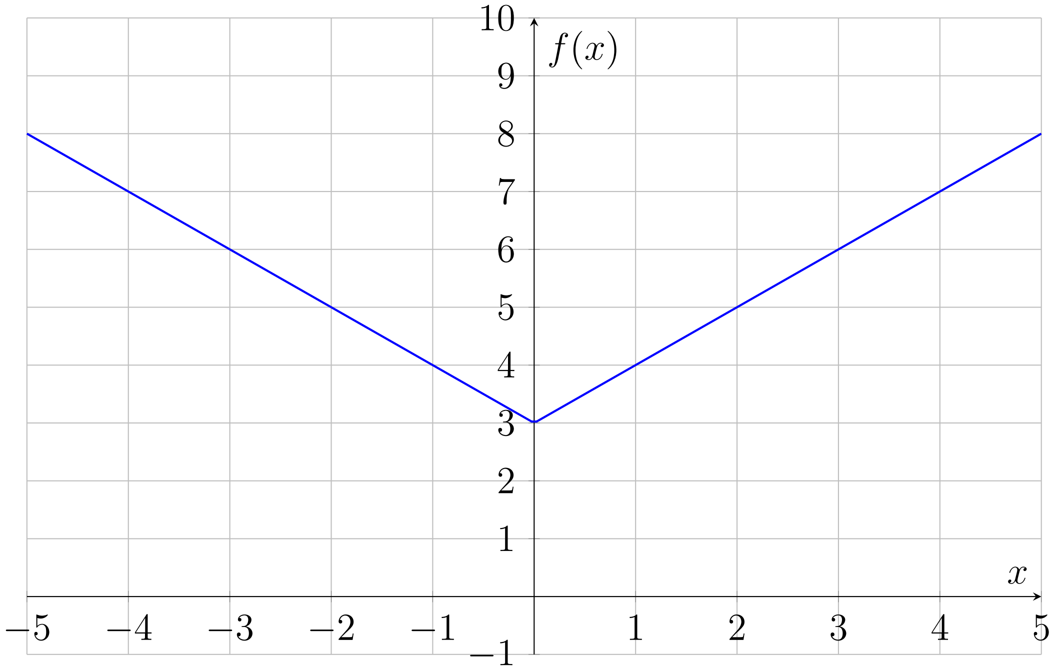
Conclusion

Since there exist horizontal lines that intersect the graph of $f(x) = -x^2 - 2$ more than once, the function is not one-to-one.

Problem 2

Graph the following function and state if it is one-to-one or not (explain how you determined this): $f(x) = |x| + 3$

Step 1: Graph the Function



Step 2: Determine if the Function is One-to-One

A function is one-to-one if and only if every horizontal line intersects the graph of the function at most once. This is known as the Horizontal Line Test.

Horizontal Line Test

For the function $f(x) = |x| + 3$, we can see that it forms a V-shape. Any horizontal line $y = c$ where $c > 3$ will intersect the graph of $f(x)$ at two points, except for the horizontal line $y = 3$ which intersects at the vertex of the V-shape.

Conclusion

Since there exist horizontal lines that intersect the graph of $f(x) = |x|+3$ more than once, the function is not one-to-one.

Problem 3

Verify if $f(x) = x^2 + 5$ and $g(x) = \sqrt{x-5}$ are inverses of each other or not.

Step 1: Verify $f(g(x)) = x$

Given:

$$f(x) = x^2 + 5$$

$$g(x) = \sqrt{x-5}$$

First, we need to find $f(g(x))$:

$$f(g(x)) = f(\sqrt{x-5})$$

Substitute $g(x)$ into $f(x)$:

$$f(\sqrt{x-5}) = (\sqrt{x-5})^2 + 5$$

Simplify:

$$f(\sqrt{x-5}) = x - 5 + 5 = x$$

Thus:

$$f(g(x)) = x$$

Step 2: Verify $g(f(x)) = x$

Next, we need to find $g(f(x))$:

$$g(f(x)) = g(x^2 + 5)$$

Substitute $f(x)$ into $g(x)$:

$$g(x^2 + 5) = \sqrt{(x^2 + 5) - 5}$$

Simplify:

$$g(x^2 + 5) = \sqrt{x^2}$$

Since $\sqrt{x^2} = x$, we have:

$$g(x^2 + 5) = x$$

Thus:

$$g(f(x)) = x$$

Conclusion

For $f(x)$ and $g(x)$ to be true inverses of each other, both $f(g(x))$ and $g(f(x))$ must equal x.

We have:

$$f(g(x)) = x$$

$$g(f(x)) = x$$

We conclude that $f(x) = x^2+5$ and $g(x) = \sqrt{x-5}$ are inverses of each other.

Problem 4

Verify if $f(x) = 2x + 1$ and $g(x) = \frac{x-1}{2}$ are inverses of each other or not.

Step 1: Verify $f(g(x)) = x$

Given:

$$f(x) = 2x + 1$$
$$g(x) = \frac{x-1}{2}$$

First, we need to find $f(g(x))$:

$$f(g(x)) = f\left(\frac{x-1}{2}\right)$$

Substitute $g(x)$ into $f(x)$:

$$f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1$$

Simplify:

$$f\left(\frac{x-1}{2}\right) = x - 1 + 1 = x$$

Thus:

$$f(g(x)) = x$$

Step 2: Verify $g(f(x)) = x$

Next, we need to find $g(f(x))$:

$$g(f(x)) = g(2x + 1)$$

Substitute $f(x)$ into $g(x)$:

$$g(2x + 1) = \frac{(2x + 1) - 1}{2}$$

Simplify:

$$g(2x+1) = \frac{2x}{2} = x$$

Thus:

$$g(f(x)) = x$$

Conclusion

Since both $f(g(x)) = x$ and $g(f(x)) = x$, we conclude that $f(x) = 2x+1$ and $g(x) = \frac{x-1}{2}$ are indeed inverses of each other.

Problem 5

Find the inverse: $f(x) = 5x + 2$

Step 1: Change $f(x)$ to y

Replace $f(x)$ with y:

$$y = 5x + 2$$

Step 2: Switch x and y

Swap x and y:

$$x = 5y + 2$$

Step 3: Solve for y

Solve the equation for y:

$$x = 5y + 2$$

Subtract 2 from both sides:

$$x - 2 = 5y$$

Divide both sides by 5:

$$y = \frac{x-2}{5}$$

Step 4: Change y to $f^{-1}(x)$

Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = \frac{x-2}{5}$$

Conclusion

The inverse function is:

$$\boxed{f^{-1}(x) = \frac{x-2}{5}}$$

Problem 6

Find the inverse: $f(x) = \frac{1}{3x-4}$

Step 1: Change $f(x)$ to y

Replace $f(x)$ with y:

$$y = \frac{1}{3x-4}$$

Step 2: Switch x and y

Swap x and y:

$$x = \frac{1}{3y-4}$$

Step 3: Solve for y

Multiply both sides by $3y - 4$

$$x(3y-4) = 1$$

Distribute:

$$3xy - 4x = 1$$

Solve for y:

$$3xy = 1 + 4x$$

$$y = \frac{1+4x}{3x}$$

Step 4: Change y to $f^{-1}(x)$

Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = \frac{1+4x}{3x}$$

Conclusion

The inverse function is:

$$\boxed{f^{-1}(x) = \frac{1+4x}{3x}}$$

Problem 7

Find the inverse: $f(x) = \frac{2x-3}{5x+7}$

Step 1: Change $f(x)$ to y

Replace $f(x)$ with y:

$$y = \frac{2x-3}{5x+7}$$

Step 2: Switch x and y

Swap x and y:

$$x = \frac{2y - 3}{5y + 7}$$

Step 3: Solve for y

Multiply both sides by $5y + 7$ to clear the fraction:

$$x(5y + 7) = 2y - 3$$

Distribute x on the left side:

$$5xy + 7x = 2y - 3$$

Move all terms involving y to one side and constant terms to the other side:

$$5xy - 2y = -7x - 3$$

Factor out y on the left side:

$$y(5x - 2) = -7x - 3$$

Divide both sides by $(5x - 2)$ to isolate y:

$$y = \frac{-7x - 3}{5x - 2}$$

Step 4: Change y to $f^{-1}(x)$

Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = \frac{-7x - 3}{5x - 2}$$

Conclusion

The inverse function is:

$$\boxed{f^{-1}(x) = \frac{-7x - 3}{5x - 2}}$$

Problem 8

Given that $f(x) = (x-1)^2$, $x \geq 1$, find $f^{-1}(x)$ and determine the domain and range of $f^{-1}(x)$.

Step 1: Change $f(x)$ to y

Replace $f(x)$ with y:

$$y = (x-1)^2$$

Step 2: Switch x and y

Swap x and y:

$$x = (y-1)^2$$

Step 3: Solve for y

Take the square root of both sides, considering $y \geq 1$:

$$\sqrt{x} = y - 1$$

Add 1 to both sides to solve for y:

$$y = \sqrt{x} + 1$$

Step 4: Change y to $f^{-1}(x)$

Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = \sqrt{x} + 1$$

Domain and Range

Recall, the domain of $f(x)$ equals the range of of f^{-1} and the range of $f(x)$ equals the domain of f^{-1}.

Domain of $f(x)$:

Given $x \geq 1$, the domain of $f(x)$ is:

$$[1, \infty)$$

Range of $f(x)$:

Since $f(x) = (x-1)^2$ and $x \geq 1$, the minimum value of $f(x)$ is 0 when $x = 1$. Thus, the range of $f(x)$ is:

$$[0, \infty)$$

Domain of $f^{-1}(x)$:

The domain of $f^{-1}(x)$ is the range of $f(x)$:

$$[0, \infty)$$

Range of $f^{-1}(x)$:

The range of $f^{-1}(x)$ is the domain of $f(x)$:

$$[1, \infty)$$

Conclusion

The inverse function is:

$$\boxed{f^{-1}(x) = \sqrt{x} + 1}$$

The domain of $f^{-1}(x)$ is:

$$\boxed{[0, \infty)}$$

The range of $f^{-1}(x)$ is:

$$\boxed{[1, \infty)}$$