Final Exam Review Worksheet, Spring 2020

1. (12 points) Indicate whether the statements are true or false.

		true	false
i)	If $A\vec{x} = \vec{b}$ has infinitely many solutions, then the RREF of A must have a row of zeros.	◯	◯
ii)	If A is $n \times n$ and $A\vec{x} = \vec{b}$ is inconsistent, then the columns of A are linearly dependent.	◯	◯
iii)	If A is a 3×3 matrix and $\det(A) = 2$, then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis for Col(A).	◯	◯
iv)	A basis for a subspace must include the zero vector.	◯	◯
v)	If the columns of an $n \times n$ matrix span $\mathbb{R}^n$, then the matrix must be invertible.	◯	◯
vi)	A matrix, A, and any echelon form of A will have the same column space.	◯	◯
xii)	An $n \times n$ diagonalizable matrix must have n distinct eigenvalues.	◯	◯
xiii)	The geometric multiplicity of an eigenvalue is greater than or equal to the algebraic multiplicity of the same eigenvalue.	◯	◯
ix)	If S is a subspace of $\mathbb{R}^8$ and $\dim(S) = 6$, then $S^{\perp}$ is a two-dimensional subspace.	◯	◯
x)	If two vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then they are linearly independent.	◯	◯
xi)	If A is symmetric, and $v_1 \neq v_2$ are two eigenvectors of A, then v_1 and v_2 are orthogonal.	◯	◯
xii)	For a symmetric matrix A, the largest value of $\|Ax\|$ subject to the constraint that $\|x\| = 1$ is the largest singular value of A.	◯	◯

You do not need to justify your reasoning for questions on this page.

2. (10 points) Fill in the blanks.

(a) List all values of $k \in \mathbb{R}$ such that the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ k \\ -1 \end{pmatrix}$ are linearly dependent. $\boxed{}$.

(b) Suppose $\det(A^2B) = 4$, $\det(B) = \frac{1}{3}$, and A and B are $n \times n$ real matrices. List all possible values of $\det(A)$. $\boxed{}$.

(c) List all values of k such that $A\vec{x} = \vec{b}$ is inconsistent where $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2k \\ 0 & 0 & k \end{pmatrix}. \quad k = \boxed{}$$

(d) Consider the row operation that reduces matrix A to RREF.

$$A = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1 A} = E_1 A$$

By inspection, E_1 is the elementary matrix $E_1 = \begin{pmatrix} \end{pmatrix}$.

(e) If $S = \{\vec{x} \in \mathbb{R}^4 \,|\, x_1 = x_2\}$ then $\dim S = \boxed{}$.

(f) If $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{pmatrix}$, then a non-zero vector in Null$A$ is $\begin{pmatrix} \end{pmatrix}$.

(g) If the basis for the column space of an 11×15 matrix consists of exactly three vectors, how many pivot columns does the matrix have? $\boxed{}$

(h) If A is a 3×3 matrix with eigenvalues 5 and $1 - i$, then the third eigenvalue is $\boxed{}$.

(i) If $\vec{v}$ is the steady-state vector for a regular stochastic matrix, then $\vec{v}$ is an eigenvector of that matrix corresponding to the eigenvalue $\lambda = \boxed{}$.

(j) List all values of k such that $A = \begin{pmatrix} 4 & k \\ 0 & 4 \end{pmatrix}$ is diagonalizable. $\boxed{}$

You do not need to justify your reasoning for questions on this page.

3. (6 points) Fill in the blanks.

(a) The distance between the vector $\vec{u} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and the line spanned by $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\boxed{}$.

(b) If W is the plane spanned by the vectors $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, a basis of $W^{\perp}$ is given by $\vec{w} = \begin{pmatrix} \end{pmatrix}$.

(c) If A is a 3×3 matrix and $\dim(\mathrm{Row}(A)) = 2$, then $\dim(\mathrm{Null}(A^T)) = \boxed{}$.

(d) If $\vec{u}$ and $\vec{v}$ are two vectors in $\mathbb{R}^2$ satisfying $\|\vec{u}\| = 3$, $\|\vec{v}\| = 2$ and $\vec{u} \cdot \vec{v} = \frac{3}{2}$, then the length of the sum of the two vectors is $\|\vec{u} + \vec{v}\| = \boxed{}$.

(e) Let U be an $n \times n$ matrix with orthonormal columns. Then $U^t U =$ ______

(f) The maximum value of $Q(\vec{x}) = 10x_1^2 - 7x_2^2 - 4x_3^2$ subject to the constraints $\vec{x} \cdot \vec{x} = 1$ and $\vec{x} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$ is equal to $\boxed{}$.

You do not need to justify your reasoning for questions on this page.

4. (8 points) Indicate whether the statements are possible or impossible.

		possible	impossible
i)	The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is onto. $T = Ax$, and A has linearly independent columns.	◯	◯
ii)	The columns of a matrix with N rows are linearly dependent and span $\mathbb{R}^N$.	◯	◯
iii)	Matrix A is $n \times n$, $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, and $\dim(\text{Null}A) = 0$.	◯	◯
iv)	P is a stochastic matrix which has zero in the first entry of the first row, but is regular.	◯	◯
v)	There is a 2×2 real matrix A and a vector $\vec{u} \neq \vec{0}$, such that $\vec{u} \in \text{Null}(A)$ and $\vec{u} \in \text{Row}(A)$.	◯	◯
vi)	A is a non-singular matrix which is not diagonalizable.	◯	◯
vi)	$\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of matrix A that correspond to distinct eigenvalues, $A = A^T$, and $\vec{v}_1 \cdot \vec{v}_2 = 1$.	◯	◯
viii)	$\vec{y}$ is a non-zero vector in $\mathbb{R}^5$. The projection of $\vec{y}$ onto a subspace of $\mathbb{R}^5$ is the zero vector.	◯	◯

5. (2 points) Suppose A and B are $n \times n$ matrices and A is symmetric. Fill in the circles next to the expressions (if any) that are equal to

$$(B^T AB)^T$$

Leave the other circles empty.

◯ $BA^T B^T$

◯ $B^T AB$

6. (2 points) List the singular values of the matrix below. (No need to justify your answer.)

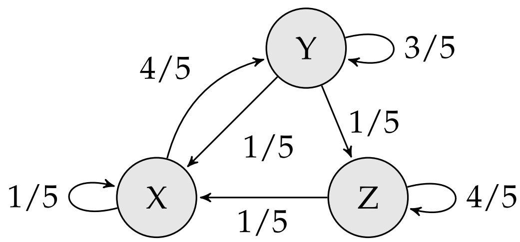
$$\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_1 = \_\_\_\_\_, \qquad \sigma_2 = \_\_\_\_\_,$$

7. (6 points) Let $A = \begin{pmatrix} -2 & -4 & 0 & 0 & 2 \\ -2 & -4 & 1 & 0 & 0 \\ -2 & -4 & 0 & 2 & 4 \\ -2 & -4 & 0 & 3 & 5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ 0 \\ 7 \\ 8 \end{pmatrix}$.

(a) Solve the system $A\vec{x} = \vec{b}$ where A and $\vec{b}$ are as above. Write your answer in parametric vector form for full credit.

(b) Write down a basis for $\mathrm{Col}(A)$.

8. (4 points) Consider the following Markov chain.



(a) What is the transition matrix, P?

$$P = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

(b) Use your transition matrix from part (a) to calculate the steady-state probability vector, $\vec{q}$. Show your work.

9. (3 points) Apply the Gram-Schmidt process to construct an orthogonal basis for $\text{Col}(A)$. Show your work.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

10. (3 points) Construct the LU factorization of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 0 \end{pmatrix}$. Clearly indicate matrices L and U.

11. (5 points) Compute Σ and V in the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} = U\Sigma V^T$$

$$\Sigma = \begin{bmatrix} \_\_\_ & 0 \\ 0 & \_\_\_ \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

12. (5 points) Find matrices D and P to construct the orthogonal diagonalization of A. Show your work.

$$A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} = PDP^T$$

$$D = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}, \qquad P = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$