Can you sit down at a piano and, nontrivially, play silence? This question lies at the heart of the HRT Conjecture which was recently shown to be false in August 2026 by mathematicians assisted by the use of AI tools. Our own Chris Heil, one of the mathematicians who originally formulated the conjecture, sits down to explain some of the background of the conjecture and to share why it was so surprising that the conjecture turned out to be false.

written by Chris Heil

A function $g(t)$ of one variable can be shifted in time by a real number a to form the new function

$$T_a g(t) = g(t-a).$$

A frequency shift of g is

$$M_b g(t) = e^{2\pi i b t} g(t).$$

A time-frequency shift is composition of these, such as

$$M_b T_a g(t) = e^{2\pi i b t} g(t-a).$$

If g is a nice bump function and you used a synthesizer to play the waveform $M_b T_a g$ through a speaker, you would hear a "note" around time a with frequency b. Loosely speaking, a single time-frequency shifts is like sitting down at a piano and hitting a single key (with frequency b) at time a. Time-frequency shifts are also closely related to coherent states in quantum mechanics, and appear widely in mathematics, physics, and engineering.

Time-frequency shifts have been widely studied. Yet in the early 1990's, Christopher Heil (then a postdoc at MIT), Jay Ramanathan, and Pankaj Topiwala (then at The MITRE Corporation) realized that one of the most fundamental questions had never been addressed: Are time-frequency shifts linearly independent? Can a nontrivial finite linear combination

$$\sum_{k=1}^{N} c_k M_{b_k} T_{a_k} g(t)$$

be the zero function? Can you sit down at a piano and, nontrivially, play silence? If we allow any functions g then time-frequency shifts can be dependent. For example, if g is the periodic function $\sin t$, then g and a single time-shift $T_{2\pi} g$ are identical and hence the collection $\{g, T_{2\pi} g\}$ is linearly dependent. However, what if we impose a reasonable restriction, namely that g has finite energy, meaning that $\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$? In that case, we say that g belongs to the Lebesgue space $L^2(\mathbb{R})$ and say that g is square-integrable.

At the time, it seemed "obvious" to those three mathematicians that if $g \in L^2(\mathbb{R})$ is not the zero function and finitely many points (a_k, b_k) in $\mathbb{R}^2$ are distinct, then the collection $\{M_{b_k} T_{a_k} g\}_{k=1,\ldots,N}$ must be linearly independent. They worked diligently to prove this and were able to establish that this is indeed true for many special cases, but could not prove independence in general. In 1996, by which time Heil had joined the School of Math at Georgia Tech, the three had published what they were able to prove, and they formally made the conjecture that time-frequency shifts of square-integrable functions must be linearly

independent. This conjecture eventually became known as the Heil-Ramanathan-Topiwala (or HRT) Conjecture, and it has since been described as one of the central problems in time–frequency analysis.

Over the 30 years since, many groups have worked on the conjecture, establishing many more partial results that seemed to support the idea that the conjecture was true. For example, the original paper established that the conjecture is true for all functions that are compactly supported (only nonzero for a finite amount of time), or for any function $g \in L^2(\mathbb{R})$ when $N = 1$, 2, or 3. Kasso Okoudjou, a Ph.D. student of Heil who graduated in 2003 (now Chair at Tufts University) made significant progress on the case $N = 4$, often in joint work with Vignon Oussa (Bridgewater State University). However, some cases for $N = 4$ remained open, even with the very strong extra assumption that g is a Schwartz-class function (a class of particularly tame infinitely differentiable functions that decay very rapidly - every function in this class and each of its derivatives tend towards zero faster than the reciprocal of *any* polynomial).

Heil says that he always believed that the conjecture is true. So it was a great surprise when in August of 2026, Markus Faulhuber, Philipp Petersen, Jordy Timo van Velthoven, and Felix Voigtlaender uploaded a preprint to the math arXiv with a counterexample for the case $N = 12$! In fact, their function g is Schwartz-class! Those authors disclose that their proof was developed with the assistance of AI (specifically ChatGPT), but it is not a proof "by AI." A few days after this remarkable announcement, Oussa announced that, using similar techniques, he was able to construct a counterexample for the case $N = 4$.

There are still many open questions. For example, the known counterexamples are all complex-valued. Indeed, Okoudjou and his Ph.D. student Shuang Guan (now a postdoc in the School of Math at Georgia Tech) recently proved that there are no real-valued counterexamples for $N = 4$. And while it is known that in some sense counterexamples are "rare," there are few quantitative formulations of that statement.