MATH 1552, INTEGRAL CALCULUS
FINAL EXAM STUDY GUIDE

In studying for the final exam, you should FIRST study all quizzes we have had this semester. Just a reminder: solutions to all quizzes can be found on Canvas. Then go over old homework problems and class worksheets.

The following list contains all the topics that will be covered on the final exam:

Riemann Sums and Average Value (5.1-5.3)

The Fundamental Theorem of Calculus (5.4)

Integration by u-Substitution (5.5)

Area Between Two Curves (5.6)

Volumes by Disks (6.1)

Integration by Parts (8.2)

Integration of Trig Functions and Powers of Trig Functions (8.3)

Integration by Trigonometric Substitution (8.4)

The Method of Partial Fractions (8.5)

Improper Integrals (8.8)

Sequences (10.1)

Geometric and Infinite Series (10.2)

Convergence Tests for Series with Nonnegative Terms (10.3-10.5)

Alternating Series Test (10.6)

Power Series (10.7)

Taylor and MacLaurin Series (10.8-10.9)

Practice Problems

Here are a few practice problems dealing with the above topics. Try to work them by yourself without using your book or notes.

Concept Review

(Note: concepts may be tested on the exam in the form of true/false or mutliple choice questions.)

(a) Properties of the definite integral:

$$\int_a^a f(x)dx =$$

$$\int_a^b cf(x)dx =$$

(b) State the Fundamental Theorem of Calculus:

(c) Using the FTC:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)dt\right] =$$

(d) If F is an antiderivative of f, that means:

(e) If F is an antiderivative of f, then:

$$\int f(g(x))g'(x)dx =$$

$$\int_a^b f(g(x))g'(x)dx =$$

(f) To find the area between two curves, use the following steps:

(g) An integral $\int_a^b f(x)dx$ is *improper* if at least one of the limits of integration is _____________, or if there is a _____________ _____________ on the interval $[a, b]$.

(h) Evaluate an integral using *integration by parts* if:

To choose the value of u, use the rule: _____________.

(i) To evaluate integrals with powers or products of trig functions, use the following trig identities to try to obtain a u-substitution:

(j) If we would evaluate an integral using *trig substitution*, the integral should contain an expression of one of these forms: _____________, _____________, or _____________.

Write out the trig substitution you would use for each form listed above.

(k) To use the method of *partial fractions*, we must first factor the denominator completely into _____________ or _____________ _____________ terms.

In the partial fraction decomposition, if the term in the denominator is raised to the kth power, then we have _____________ partial fractions.

For each linear term, the numerator of the partial fraction will be ____________.

For each irreducible quadratic term, the numerator will be ____________.

(l) A geometric series has the general form ____________.
The series converges when ____________ and diverges when ____________.

(m) A p-series has the general form ____________. The series converges when ____________ and diverges when ____________. To show these results, we can use the ____________ test.

(n) The harmonic series is the series with formula: ____________. Does it converge or diverge?

(o) If you want to show a series converges, compare it to a ____________ series that also converges. If you want to show a series diverges, compare it to a ____________ series that also diverges.

(p) If the direct comparison test does not have the correct inequality, you can instead use the ____________ ____________ test. In this test, if the limit is a ____________ number (not equal to ____________), then both series converge or both series diverge.

(q) In the ratio and root tests, the series will ____________ if the limit is less than 1 and ____________ if the limit is greater than 1. If the limit equals 1, then the test is ____________.

(r) If $\lim_{n\to\infty} a_n = 0$, then what do we know about the series $\sum_n a_n$? ____________

(s) A power series has the general form: ____________. To find the radius of convergence R, use either the ____________ or ____________ test. The series converges ____________ when $|x - c| < R$. To find the interval of convergence, don't forget to check the ____________.

(t) If $\sum_k a_k$ is an alternating series, then it converges ____________ if $\sum_k |a_k|$ converges. It converges ____________ if $\sum_k |a_k|$ diverges and (i) the limit of the terms is ____________ and (ii) the sequence of terms is ____________.

(u) If an alternating series converges, we can estimate the sum by adding the first n terms. Stopping after n terms will give us an error at most equal to the magnitute of the ____________ term in the sequence.

(v) A Taylor polynomial has the general form: ____________. The Taylor polynomial is the n^{th} ____________ ____________ of the Taylor series with general form: ____________.

(w) The Taylor remainder theorem says that $|R_n| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$, where M represents the maximum value of the ____________ derivative of f on the interval between x and a. The remainder term decreases when n ____________ or when x is ____________ to a.

(z) A MacLaurin Series is a Taylor series centered at ____________.

(aa) Complete the formulas for the common MacLaurin series.

$$e^x = \sum_{k=0}^{\infty}$$

$$\ln(1+x) = \sum_{k=0}^{\infty}$$

$$\sin(x) = \sum_{k=0}^{\infty}$$

$$\cos(x) = \sum_{k=0}^{\infty}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty}$$

(bb) Fill in the formulas for the derivatives and anti-derivatives of a power series.

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k x^k\right] =$$

$$\int_0^x \left[\sum_{k=0}^{\infty} a_k t^k\right] dt =$$

(cc) Fill in the integration formulas below:

$$\int x^n dx, \quad (n \neq -1) =$$

$$\int \sin(ax) dx =$$

$$\int \cos(ax) dx =$$

$$\int \sec^2(ax)dx =$$

$$\int \sec(ax)\tan(ax)dx =$$

$$\int \csc(ax)\cot(ax)dx =$$

$$\int \csc^2(ax)dx =$$

$$\int \frac{1}{1+(ax)^2}dx =$$

$$\int \frac{1}{\sqrt{1-(ax)^2}}dx =$$

$$\int \frac{1}{x}dx =$$

$$\int e^{ax}dx =$$

$$\int b^{ax}dx =$$

$$\int \tan x dx =$$

$$\int \sec x dx =$$

$$\int \csc x dx =$$

$$\int \cot x dx =$$

PRACTICE PROBLEMS

1. Find $F'(2)$ for the function

$$F(x) = \int_{\frac{8}{x}}^{x^2} \left(\frac{t}{1-\sqrt{t}}\right) dt.$$

2. Let $\{a_n\}$ and $\{b_n\}$ be a sequences of non-negative terms. Are the following statements *always* true or sometimes false?

(a) If $\lim_{n\to\infty} a_n = L$, then the series $\sum_n a_n = L$.

(b) If $\lim_{n\to\infty} a_n = 0$, then $\{a_n\}$ converges to 0.

(c) If $\lim_{n\to\infty} a_n = 0$, then $\sum_n a_n$ converges.

(d) If $\sum_n a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(e) If $\sum_n a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

(f) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_n a_n$ diverges.

(g) If $\lim_{n\to\infty} a_n \neq 0$, then $\{a_n\}$ diverges.

(h) If $\int_1^\infty f(x)dx = L$, where $0 < L < \infty$, then $\sum_n f(n) = L$.

(i) If $\sum_n b_n$ converges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also converges.

(j) If $\sum_n b_n$ diverges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also diverges.

(k) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_n b_n$ diverges, then $\sum_n a_n$ also diverges.

3. Find the l.u.b. and g.l.b. of the following sequences.

(a) $\left\{\frac{2n}{n+1}\right\}$

(b) $\left\{1 - \frac{2}{n}\right\}$

4. Determine whether the following sequences converge. If so, to what? If not, why not?

(a) $\left\{\frac{n!}{n}\right\}$

(b) $\left\{\frac{5n^2-4}{3-7n^2}\right\}$

5. Evaluate the following integrals.

(a) $\int \frac{x^4+x^6+1}{x^2} dx$

(b) $\int_1^2 \frac{\sqrt{1+\ln(x)}}{x} dx$

(c) $\int_1^e \frac{2\ln(x^x)+1}{x^2} dx$

(d) $\int e^x \ln(e^x+1)dx$

(e) $\int \tan(x)\ln|\cos(x)|dx$

(f) $\int \frac{x+4}{x^3+3x^2-10x} dx$

(g) $\int \tan^3(x)\sec^3(x)dx$

(h) $\int \frac{x^2}{\sqrt{4-x^6}} dx$

(i) $\int_0^2 x|2x-1|dx$

(j) $\int \sin^4(3x)dx$

(k) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

(l) $\int x^5 \sin(x^3)dx$

(m) $\int \frac{dx}{\sqrt{e^{2x}-9}}$

(n) $\int 5^x \cos(2x)dx$

6. Find the area of the region bounded by the curves $y = 5x + 1$ and $y = x^2 + 3x - 2$.

7. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin(x)$, the x-axis, and the lines $x = 0$, $x = \pi/2$ about the x-axis.

8. Find the volume of the solid generated by rotating the region bounded by $y = x^3$, $x = 1$, and $y = -1$ about the line $y = -1$.

9. Find the volume of the solid generated when rotating the region:
(a) bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ about the x-axis.

(b) bounded by the curves $y = x^2 - 4$ and $y = 2x - x^2$ about the line $y = -4$.

10. Determine if the following series converge or diverge using the convergence tests from class.
(a) $\sum_{k=2}^{\infty}(-1)^k \frac{1}{k \ln k \sqrt{\ln \ln k}}$
(b) $\sum_{k=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2k-1)}{k3^{k+1}k!}$
(c) $\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$
(d) $\sum_{k=1}^{\infty} \frac{(k^2+1)^{1/3}}{\sqrt{k^3+3}}$
(e) $\sum_{k=1}^{\infty}(1 - \frac{3}{k})^{k^2}$
(f) $\sum_{k=4}^{\infty} \frac{(-1)^k \sin(1/k)}{k^2}$

11. Find the radius and interval of convergence of the following power series:
(a) $\sum_{k=2}^{\infty} \left(\frac{k}{k-1}\right) \frac{(x+2)^k}{2^k}$

(b) $\sum_{k=1}^{\infty} \frac{k}{3^k(k^2+1)}(x-5)^k$

(c) $\sum_{k=1}^{\infty} \frac{5^k}{\sqrt{k}}(3-2x)^k$

(d) $\sum_{k=1}^{\infty} \frac{(-1)^k a^k}{k^2}(x-a)^k$, where $a \neq 0$

12. Consider the power series $f(x) = \sum_{n=0}^{\infty} \left(\frac{x^2+2}{6}\right)^n$.

(a) Find the interval of convergence for this series.

(b) For all values of x that lie in the interval of convergence, what is the sum of this series as a function of x?

13. Determine how many terms must be used to determine the sum of the entire series with an error of less than 0.01

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+1}$

(b) $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n$

(c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

14. Find a power series representation for the following functions. When is your series valid?

(a) $f(x) = \frac{3x}{2+4x}$

(b) $g(x) = \int_0^x \frac{\sin(t/2)}{2t}dt$

(c) $h(x) = \tan^{-1}(x)$

15. (a) Estimate $\cos 15°$ using a fourth-degree Taylor polynomial.

(b) Estimate $\int_0^1 e^{-2x^2}dx$ within an error of 0.01.

16. Sum the following series.

(a) $\sum_{k=3}^{\infty}\frac{3^{k-2}+2^{k+1}}{5^{k+1}}$

(b) $\sum_{k=4}^{\infty}\frac{1}{(3k+1)(3k+4)}$

Answers to Practice Problems

1. -24

3. (a) l.u.b. = 2, g.l.b. = 1; (b) l.u.b. = 1; g.l.b. = -1

4. (a) Diverges (since $\lim_{n\to\infty} a_n = \infty$);

(b) Converges to $-\frac{5}{7}$

5. (a) $\frac{x^3}{3} + \frac{x^5}{5} - \frac{1}{x} + C$

(b) $\frac{2}{3}(1 + \ln(2))^{3/2} - \frac{2}{3}$

(c) $2 - \frac{1}{e}$

(d) $(e^x + 1)\ln(e^x + 1) - (e^x + 1) + C$

(e) $-\frac{1}{2}(\ln|\cos(x)|)^2$ + C

(f) $-\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C$

(g) $\frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C$

(h) $\frac{1}{3}\sin^{-1}\left(\frac{x^3}{2}\right) + C$

(i) $\frac{41}{12}$

(j) $\frac{3}{8}x - \frac{1}{12}\sin(6x) + \frac{1}{96}\sin(12x) + C$

(k) $-\frac{\sqrt{1-x^2}}{x} + C$

(l) $-\frac{1}{3}x^3\cos(x^3) + \frac{1}{3}\sin(x^3) + C$

(m) $\frac{1}{3}\sec^{-1}\left(\frac{e^x}{3}\right) + C$

(n) $\frac{\ln 5}{4+(\ln 5)^2}5^x\cos(2x) + \frac{2}{4+(\ln 5)^2}5^x\sin(2x) + C$

6. $\frac{32}{3}$

7. $\frac{\pi^2}{4}$

8. $\frac{16}{7}\pi$

9. (a) $\frac{108\pi}{5}$, (b) 45π, (c) $\frac{64\pi\sqrt{2}}{15}$, (d) $\frac{\pi^2}{16}$ (all in cubic units)

10. (a) converges conditionally; (b) and (e) converge; (c) and (d) diverge; (f) converges absolutely

11. (a) $R = 2$, $I.C. = (-4, 0)$, (b) $R = 3$, $I.C. = [2, 8)$ (c) $R = \frac{1}{10}$, $I.C. = \left(\frac{7}{5}, \frac{8}{5}\right]$, (d) $R = \frac{1}{|a|}$, $I.C. = [a - \frac{1}{|a|}, a + \frac{1}{|a|}]$

12. $I.C. = (-2, 2)$, sum is $\frac{6}{4-x^2}$

13. (a) 9, (b) 4, (c) 100

14. (a) $3\sum_{k=0}^{\infty}(-1)^k 2^{k-1} x^{k+1}$, valid for $|x| < \frac{1}{2}$
(b) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2^{2k+2}(2k+1)!(2k+1)}$, valid for $x \neq 0$
(c) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} + C$, valid for $|x| < 1$

15. (a) 0.966; (b) approximately 0.6165 (stop at $k = 5$)

16. (a) $\frac{41}{750}$; (b) $\frac{1}{39}$