Math 1552: Integral Calculus

Final Exam Study Guide, Spring 2018

PART 1: Concept Review

(Note: concepts may be tested on the exam in the form of true/false or short-answer questions.)

1. Complete each statement below.

(a) Properties of the definite integral:

$$\int_a^a f(x)dx =$$

$$\int_a^b cf(x)dx =$$

(b) State the Fundamental Theorem of Calculus:

(c) Using the FTC:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)dt\right] =$$

(d) If F is an antiderivative of f, that means:

(e) If F is an antiderivative of f, then:

$$\int f(g(x))g'(x)dx =$$

$$\int_a^b f(g(x))g'(x)dx =$$

(f) To find the area between two curves, use the following steps:

(g) A *separable differential equation* has the general form:

To solve this equation, use the following steps:

(h) An integral $\int_a^b f(x)dx$ is *improper* if at least one of the limits of integration is ____________, or if there is a ____________ ____________ on the interval $[a, b]$.

(i) Evaluate an integral using *integration by parts* if:

To choose the value of u, use the rule: ____________.

(j) To evaluate integrals with powers or products of trig functions, use the following trig identities to try to obtain a u-substitution:

(k) If we would evaluate an integral using *trig substitution*, the integral should contain an expression of one of these forms: ____________, ____________, or ____________.

Write out the trig substitution you would use for each form listed above.

(l) To use the method of *partial fractions*, we must first factor the denominator completely into ____________ or ____________ ____________ terms.

In the partial fraction decomposition, if the term in the denominator is raised to the kth power, then we have ____________ partial fractions.

For each linear term, the numerator of the partial fraction will be ____________.

For each irreducible quadratic term, the numerator will be ____________.

(m) A geometric series has the general form ____________.
The series converges when ____________ and diverges when ____________.

(n) A p-series has the general form ____________. The series converges when ____________ and diverges when ____________. To show these results, we can use the ____________ test.

(o) The harmonic series is the series with formula: ____________. Does it converge or diverge?

(p) If you want to show a series converges, compare it to a ____________ series that also converges. If you want to show a series diverges, compare it to a ____________ series that also diverges.

(q) If the direct comparison test does not have the correct inequality, you can instead use the ____________ ____________ test. In this test, if the limit is a ____________ number (not equal to ____________), then both series converge or both series diverge.

(r) In the ratio and root tests, the series will ____________ if the limit is less than 1 and ____________ if the limit is greater than 1. If the limit equals 1, then the test is ____________.

(s) If $\lim_{n\to\infty} a_n = 0$, then what do we know about the series $\sum_n a_n$? ____________

(t) A power series has the general form: ____________. To find the radius of convergence R, use either the ____________ or ____________ test. The series converges ____________ when $|x - c| < R$. To find the interval of convergence, don't forget to check the ____________.

2. Fill in the integration formulas below:

$$\int x^n dx, \quad (n \neq -1) =$$

$$\int \sin(ax)dx =$$

$$\int \cos(ax)dx =$$

$$\int \sec^2(ax)dx =$$

$$\int \sec(ax)\tan(ax)dx =$$

$$\int \csc(ax)\cot(ax)dx =$$

$$\int \csc^2(ax)dx =$$

$$\int \frac{1}{1+(ax)^2}dx =$$

$$\int \frac{1}{\sqrt{1-(ax)^2}}dx =$$

$$\int \frac{1}{x}dx =$$

$$\int e^{ax}dx =$$

$$\int b^{ax}dx =$$

$$\int \tan x dx =$$

$$\int \sec x dx =$$

$$\int \csc x dx =$$

$$\int \cot x dx =$$

PART 2: Problems from Test 1 Study Guide

1. Find $F'(2)$ for the function

$$F(x) = \int_{\frac{8}{x}}^{x^2} \left(\frac{t}{1-\sqrt{t}} \right) dt.$$

2. Let $y = \int_{-4}^{\tan x} \sin(t^2) dt$. Find $\frac{dy}{dx}$ for $0 < x < \pi/4$.

3. Evaluate the integrals using the FTC or properties of integration.

(a) $\int_1^2 \frac{3x-5}{x^3} dx.$

(b) $\int_1^3 |x-2| dx.$

(c) $\int_{\pi/2}^{\pi} \left(\frac{1}{x^2} + 2\sqrt{x} + \cos x \right) dx.$

(d) $\int_{-3}^{3} \sqrt{9-x^2} dx$

(e) $\int_{-3}^{3} \sin(x^5) dx$

(f) $\int_{-1}^{2} |2x| dx$

(a) $\int \left(\sqrt{x} - \frac{1}{x^2} \right)^2 dx$

4. Evaluate the following integrals.

(a) $\int \frac{1}{x^2} \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) dx$

(b) $\int \frac{\sin 3x - \cos 3x}{\sin 3x + \cos 3x} dx$

(c) $\int \frac{1}{\ln(x^x)} dx$

(d) $\int \frac{e^{2x}}{\sqrt{4-3e^{2x}}} dx$

(e) $\int \frac{dx}{\sqrt{4-(x+3)^2}}$

(f) $\int \frac{\log_3 x^4}{x} dx$

(g) $\int_1^e \frac{\sqrt{\ln x}}{x} dx$

(h) $\int_{-5}^{0} \left(x\sqrt{4-x} \right) dx$

(i) $\int (1+\ln x) \cot(x \ln x) dx$

(j) $\int \frac{6}{\sqrt{y}(5+6\sqrt{y})^5} dy$

(k) $\int \frac{1}{\sqrt{x}e^{-\sqrt{x}}} \csc^2\left(3e^{\sqrt{x}} - 3\right) dx$

(l) $\int \frac{1}{\sin^{-1}(x)\sqrt{1-x^2}} dx$

5. Find the area of each region.

(a) bounded by the curves $y = -x^3 - 2x^2 + 7x - 2$ and $y = -x - 2$.

(b) bounded by the curves $f(x) = x^3 + 2x^2$ and $g(x) = x^2 + 2x$.

(c) enclosed by the three curves $y = x^3$, $y = -x$, and $y = -1$.

(d) enclosed by the triangle with vertices at the points (0,1), (3,4), and (4,2). USE CALCULUS.

ANSWERS TO PART 2

1. -24

2. $\sin\left(\tan^2 x\right) \cdot \sec^2 x$

3. (a) $-\frac{3}{8}$, (b) 1,

(c) $-\frac{1}{\pi} + 4/3\sqrt{\pi^3} - \left(-\frac{1}{\pi/2} + 4/3\sqrt{(\pi/2)^3} + 1\right)$,

(d) $\frac{9}{2}\pi$, (e) 0, (f) 5,

(g) $\frac{1}{2}x^2 + \frac{4}{\sqrt{x}} - \frac{1}{3x^3} + C$

4. (a) $-\sec\left(\frac{1}{x}\right) + C$

(b) $-\frac{1}{3}\ln|\sin 3x + \cos 3x| + C$

(c) $\ln|\ln x| + C$

(d) $-\frac{1}{3}\sqrt{4 - 3e^{2x}}$

(e) $\sin^{-1}\left(\frac{x+3}{2}\right) + C$

(f) $2\ln 3(\log_3 x)^2 + C$

(g) $\frac{2}{3}$

(h) $-\frac{506}{15}$

(i) $\ln|\sin(x\ln x)| + C$

(j) $-\frac{1}{2(5+6\sqrt{y})^4} + C$

(k) $-\frac{2}{3}\cot\left(3e^{\sqrt{x}} - 3\right) + C$

(l) $\ln\left(\sin^{-1}(x)\right) + C$

5. (a) $\frac{148}{3}$ square units, (b) $\frac{37}{12}$ square units, (c) $\frac{5}{4}$ square units, (d) 4.5 square units

PART 3: Problems from Test 2 Study Guide

1. Solve the initial value problem $y' = xy - x - y + 1$, $y(1) = 0$.

2. Evaluate each improper integral if it converges, or show that the integral diverges.

(a) $\int_0^3 \frac{x}{(x^2-1)^{2/3}} dx$

(b) $\int_1^3 \frac{1}{(x^2-1)^{3/2}} dx$

(c) $\int_0^\infty x^2 e^{-2x} dx$

3. Find the area bounded by the curve $y = \frac{1}{x^2+9}$, the x-axis, and $x \geq 0$.

4. Evaluate each integral below using any of the methods we have learned: u-substitution, integration by parts, trigonometric identities, trig substitution, or partial fractions.

(a) $\int \frac{\sin^3 x}{\cos x} dx$

(b) $\int 3x \cos(2x) dx$

(c) $\int x^5 \ln(x) dx$

(d) $\int x^3 e^{x^2} dx$

(e) $\int (\ln x)^2 dx$

(f) $\int x^2 \cdot 4^x dx$

(g) $\int \cos(2x) e^x dx$

(h) $\int \sin^5(2x) \cos^3(2x) dx$

(i) $\int_1^e \frac{\sqrt{\ln x}}{x} dx$

(j) $\int \frac{x^2}{(x^2+4)^{3/2}} dx$

(k) $\int \frac{x}{(4-x^2)^{3/2}} dx$

(l) $\int \frac{dx}{e^x \sqrt{e^{2x}-9}}$

(m) $\int \sin^2(x) \cos^2(x) dx$

(n) $\int \frac{x+3}{(x-1)(x^2-4x+4)} dx$

(o) $\int \frac{x+4}{x^3+x}dx$

(p) $\int \frac{x+2}{x+1}dx$

(q) $\int \sqrt{25-x^2}dx$

(r) $\int x\tan^{-1}(x)dx$

(s) $\int \frac{dx}{x\sqrt{1+x^2}}$

(t) $\int \frac{\sqrt{1-x^2}}{x^2}\ dx$

(u) $\int \frac{1}{\sqrt{(1+x^2)^3}}\ dx$

(v) $\int \tan^3(x)\sec^3(x)\ dx$

(w) $\int \sec^4(x)\ dx$

(x) $\int \frac{\cos(2x)}{\sin^2(2x)-3\sin(2x)-4}dx$

ANSWERS TO PART 3

1. $y = -e^{(x-1)^2/2} + 1$ or $y = -\sqrt{e} \cdot e^{x^2/2 - x} + 1$

2. (a) Converges to $\frac{9}{2}$, (b) diverges, (c) converges to $\frac{1}{4}$

3. $\frac{\pi}{6}$ units^2

4. (a) $-\ln|\cos x| + \frac{1}{2}\cos^2 x + C$

(b) $\frac{3}{2}x\sin(2x) + \frac{3}{4}\cos(2x) + C$

(c) $\frac{x^6 \ln x}{6} - \frac{x^6}{36} + C$

(d) $\frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$

(e) $x(\ln x)^2 - 2x\ln x + 2x + C$

(f) $\frac{1}{\ln 4}x^2 \cdot 4^x - \frac{2}{(\ln 4)^2}x \cdot 4^x + \frac{2}{(\ln 4)^3}4^x + C$

(g) $\frac{1}{5}\cos(2x)e^x + \frac{2}{5}\sin(2x)e^x + C$

(h) $\frac{1}{12}\sin^6(2x) - \frac{1}{16}\sin^8(2x) + C$

(i) $\frac{2}{3}$

(j) $\ln|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}| - \frac{x}{\sqrt{x^2+4}} + C$

(k) $\frac{1}{\sqrt{4-x^2}} + C$

(l) $\frac{\sqrt{e^{2x}-9}}{9e^x} + C$

(m) $\frac{x}{8} - \frac{1}{32}\sin(4x) + C$

(n) $4\ln\left|\frac{x-1}{x-2}\right| - \frac{5}{x-2} + C$

(o) $4\ln|x| - 2\ln(x^2+1) + \tan^{-1}(x) + C$

(p) $x + \ln|x+1| + C$

(q) $\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{2} + C$

(r) $\frac{x^2}{2}\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$

(s) $-\ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + C$

(t) $-\frac{\sqrt{1-x^2}}{x} - \sin^{-1} x + C$

(u) $\frac{x}{\sqrt{1+x^2}} + C$

(v) $-\frac{1}{3}\sec^3(x)+\frac{1}{5}\sec^5(x)+C$

(w) $(2\tan(x))/3+1/3\sec(x)^2\tan(x)+C$

(x) $-\frac{1}{10}\ln|\sin(2x)+1|+\frac{1}{10}\ln|\sin(2x)-4|+C$

PART 4: Problems from Test 3 Study Guide

1. Let $\{a_n\}$ and $\{b_n\}$ be a sequences of non-negative terms. Are the following statements *always* true or sometimes false?

(a) If $\lim_{n\to\infty} a_n = L$, then the series $\sum_n a_n = L$.

(b) If $\lim_{n\to\infty} a_n = 0$, then $\{a_n\}$ converges to 0.

(c) If $\lim_{n\to\infty} a_n = 0$, then $\sum_n a_n$ converges.

(d) If $\sum_n a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(e) If $\sum_n a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

(f) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_n a_n$ diverges.

(g) If $\lim_{n\to\infty} a_n \neq 0$, then $\{a_n\}$ diverges.

(h) If $\int_1^\infty f(x)dx = L$, where $0 < L < \infty$, then $\sum_n f(n) = L$.

(i) If $\sum_n b_n$ converges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also converges.

(j) If $\sum_n b_n$ diverges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also diverges.

(k) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_n b_n$ diverges, then $\sum_n a_n$ also diverges.

2. **Find the sum** of each convergent series below, or explain why the series diverges.

(a) $\sum_{k=7}^{\infty} \frac{1}{(k-3)(k+1)}$

(b) $\sum_{k=0}^{\infty} (-1)^k$

(c) $\sum_{k=2}^{\infty} \frac{2^k+1}{3^{k+1}}$

3. Determine whether or not each sequence converges. If so, find the limit.

(a) $\left\{\frac{2n^2}{\sqrt{9n^4+1}}\right\}$

(b) $\left\{\left(1-\frac{1}{8n}\right)^n\right\}$

(c) $\left\{\frac{n!}{e^n}\right\}$

(d) $\left\{\left(\frac{n}{n+5}\right)^n\right\}$

4. Determine if each series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using any of the convergence tests from class.

(a) $\sum_{k=1}^{\infty} \frac{e^k}{4+e^{2k}}$

(b) $\sum_{k=1}^{\infty} \frac{5k^2+8}{7k^2+6k+1}$

(c) $\sum_{k=1}^{\infty} \frac{3^{2k}}{8^k-3}$

(d) $\sum_{k=1}^{\infty} \frac{k+2}{\sqrt{k^5+4}}$

(e) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ...$

(f) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

(g) $\sum_{k=1}^{\infty} \frac{\ln k}{k^4}$

(h) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{4^n 2^n n!}$

(i) $\sum_{k=1}^{\infty} \frac{(2k)^k}{k!}$

(j) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{2k^2}$

(k) $\sum_{n=2}^{\infty} \frac{2-\cos(n)}{n^2-1}$

(l) $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

5. Determine whether the following alternating series converge absolutely, converge conditionally, or diverge. Justify your answers using the tests we discussed in class.

(a) $\sum_{k=2}^{\infty}(-1)^{k+1} \frac{3k}{\sqrt{k^3+4}}$

(b) $\sum_{k=2}^{\infty}(-1)^{k} \frac{k}{k^4-1}$

(c) $\sum_{k=0}^{\infty}(-1)^{k} \frac{k}{5^k+2^k}$

(d) $\sum_{n=1}^{\infty}(-1)^{n} \frac{n!}{n^n}$

(e) $\sum_{n=1}^{\infty}(-1)^{3n+1} \frac{(1+5n^4)^n}{(1+3n^2)^{2n}}$

(f) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

(g) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{(n+1)^2}$

6. Find the radius and interval of convergence of the following power series:

(a) $\sum_{k=2}^{\infty} \left(\frac{k}{k-1}\right) \frac{(x+2)^k}{2^k}$

(b) $\sum_{k=1}^{\infty} \frac{k}{3^k(k^2+1)}(x-5)^k$

(c) $\sum_{k=1}^{\infty} \frac{5^k}{\sqrt{k}}(3-2x)^k$

(d) $\sum_{n=0}^{\infty} \frac{3^{2n-1}x^n}{4^n}$

7. (a) Give an example of a divergent monotonic sequence.

(b) Give an example of a divergent bounded sequence.

8. For what values of p does the series $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

9. Let $a_n = \ln\left(\frac{n+1}{n}\right)$.

(a) Does the sequence $\{a_n\}$ converge or diverge? If it converges, find the limit.

(b) Find a formula for the n-th partial sum $s_n = a_1 + a_2 + \cdots + a_n$. Simplify.

(c) Does the series $\sum_{k=1}^{\infty} a_k$ converge or diverge? If it converges, find the sum.

10. (a) Does the series $\sum_{n=2}^{\infty} \frac{(\pi-1)^{n+1}}{e^{n-1}}$ converge or diverge? If it converges, find the sum.

(b) Does the series $\sum_{n=1}^{\infty} 3^{-\frac{1}{n}}$ converge or diverge?

ANSWERS TO PART 4

1. Statements (b), (d), (f), and (j) are true.

2. (a) $\frac{319}{1,680}$, (b) diverges, (c) $\frac{1}{2}$

3. (a) converges to $\frac{2}{3}$, (b) converges to $e^{-1/8}$, (c) diverges, (d) converges to $\frac{1}{e^5}$

4. (a) converges by integral test, (b) diverges by divergence test, (c) diverges by direct comparison, (d) converges by direct or limit comparison, (e) converges by direct comparison, (f) converges by limit comparison, (g) converges by direct comparison, (h) converges by ratio test, (i) diverges by ratio test, (j) converges by root test, (k) converges by direct comparison, (l) converges by integral test

5. (a) converges conditionally (limit comparison and alternating series test), (b) converges absolutely (limit comparison), (c) converges absolutely (ratio test), (d) converges absolutely (use ratio test); (e) converges absolutely (use root test) ; (f) converges absolutely (use integral test); (g) converges conditionally (use limit comparison)

6. (a) $R = 2$, $I.C. = (-4, 0)$, (b) $R = 3$, $I.C. = [2, 8)$ (c) $R = \frac{1}{10}$, $I.C. = \left(\frac{7}{5}, \frac{8}{5}\right]$, (d) $R = \frac{4}{9}$, $I.C. = \left(-\frac{4}{9}, \frac{4}{9}\right)$,

7. answers may vary

8. converges when $p > 1$

9. (a) converges to 0; (b) $s_n = \ln(n + 1)$; (c) diverges

10. (a) converges to $\frac{(\pi-1)^3}{e-\pi+1}$; (b) diverges by the n^{th} term test

PART 5: Problems from Sections 10.8-10.9, 6.1-6.2

1. Find the third degree Taylor polynomial of the function $f(x) = \tan^{-1}(x)$ in powers of $x - 1$.

2. Use a Taylor polynomial to estimate the value of $\sqrt{e}$ with an error of at most 0.01. HINT: Choose $a = 0$ and use the fact that $e < 3$.

3. Use the MacLaurin series for $f(x) = \frac{1}{1-x}$ to find a power series representation of the function

$$g(x) = \frac{x}{(1-x)^3}.$$

HINT: You will need to differentiate.

4. Find $f^{(7)}(0)$ for the function $f(x) = x\sin(x^2)$.

5. Let $f(x) = \int_0^x t\sin(t^3)dt$. Use a MacLaurin series to find $f^{(11)}(0)$.

6. Find the sum of the series:

$$1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} + ... + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + ...$$

7. For what values of x can we replace $\cos x$ with $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ within an error range of no more that 0.001?

8. Find a power series representation for the following functions. When is your series valid?

(a) $f(x) = \frac{3x}{2+4x}$

(b) $g(x) = \int_0^x \frac{\sin(t/2)}{2t}dt$

(c) $h(x) = \tan^{-1}(x)$

9. (a) Estimate $\cos\left(\frac{\pi}{12}\right)$ using a fourth-degree Taylor polynomial.

(b) Estimate $\int_0^1 e^{-2x^2}dx$ within an error of 0.01.

10. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin(x)$, the x-axis, and the lines $x = 0$, $x = \pi/2$ about the y-axis.

11. Find the volume of the solid generated when the region bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ is revolved about the x-axis.

12. Find the volume of the solid generated when the region bounded by the curves $y = x^2 - 4$ and $y = 2x - x^2$ is revolved about the line (a) $y = -4$ and (b) $x = 2$.

13. Use the method of cylindrical shells to find the volume of the solid generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 9$ is revolved about the x-axis.

ANSWERS TO PART 5

1. $P_3(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$

2. $\sqrt{e} \approx f(0.5) = 1 + 0.5 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} = 1.6458.$

3. $\frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-1}$, $|x| < 1$

4. -840

5. $-\frac{10!}{6}$

6. $\frac{1}{2}$

7. $x \in (-0.9467, 0.9467)$

8. (a) $3\sum_{k=0}^{\infty}(-1)^k 2^{k-1}x^{k+1}$, valid for $|x| < \frac{1}{2}$
(b) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2^{2k+2}(2k+1)!(2k+1)}$, valid for $x \neq 0$
(c) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$, valid for $|x| < 1$

9. (a) 0.966; (b) approximately 0.6165 (stop at $k = 5$)

10. 2π

11. $\frac{108\pi}{5}$ cubic units

12. (a) 45π cubic units, (b) 27π cubic units

13. $\frac{81\pi}{2}$ cubic units