- - - - - - - - - - - - - - -
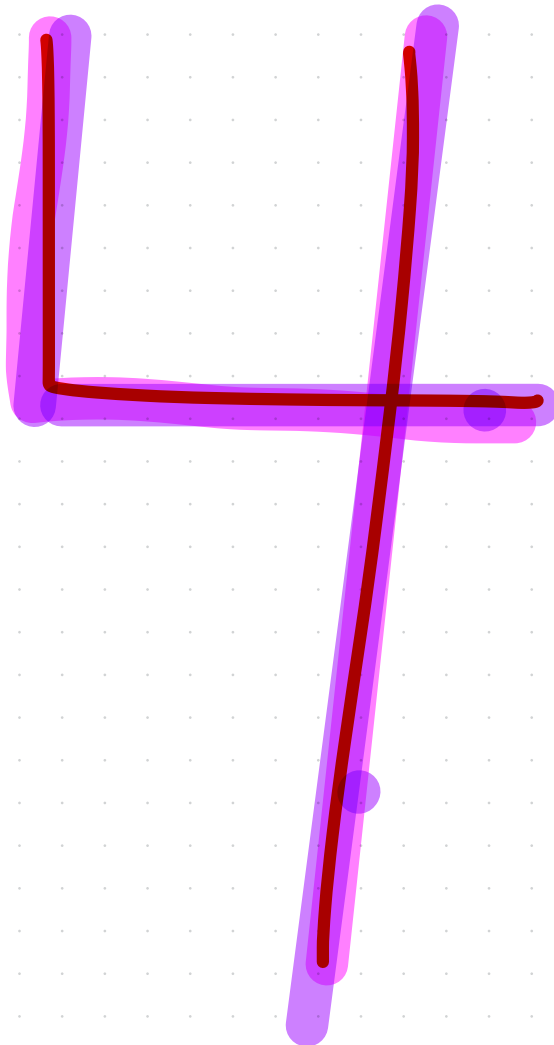UNIT 4 - Trigonometric formulas and equations

5.1 trigonometric identities, pythagorus, even/odd trig functions
5.2 sum and difference formula for trig functions
5.3 double angle formula, half angle formula
5.4 sum-to-product and product-to-sum formulas
5.5 solving trigonometric equations

Building Skills

In Exercises 9–18, use the fundamental identities and appropriate algebraic operations to simplify each expression.

9. $(1 + \tan x)(1 - \tan x) + \sec^2 x$

10. $(\sec x - 1)(\sec x + 1) - \tan^2 x$

11. $(\sec x + \tan x)(\sec x - \tan x)$

12. $\dfrac{\sec^2 x - 4}{\sec x - 2}$

13. $\csc^4 x - \cot^4 x$

14. $\sin x \cos x (\tan x + \cot x)$

15. $\dfrac{\sec x \csc x (\sin x + \cos x)}{\sec x + \csc x}$

16. $\dfrac{1}{\csc x + 1} - \dfrac{1}{\csc x - 1}$

17. $\dfrac{\tan^2 x - 2 \tan x - 3}{\tan x + 1}$

18. $\dfrac{\tan^2 x + \sec x - 1}{\sec x - 1}$

In Exercises 19–24, prove that the given equation is not an identity by finding a value of x for which the two sides have different values. The answers may vary.

19. $\sin x = 1 - \cos x$

20. $\tan x = \sec x - 1$

21. $\cos x = \sqrt{1 - \sin^2 x}$

22. $\sec x = \sqrt{1 + \tan^2 x}$

23. $\sin^2 x = (1 - \cos x)^2$

24. $\cot^2 x = (\csc x + 1)^2$

In Exercises 25–68, verify each identity.

25. $\sin x \tan x + \cos x = \sec x$

26. $\cos x \cot x + \sin x = \csc x$

27. $\dfrac{1 - 4\cos^2 x}{1 - 2\cos x} = 1 + 2\cos x$

28. $\dfrac{9 - 16 \sin^2 x}{3 + 4 \sin x} = 3 - 4 \sin x$

29. $(\cos x - \sin x)(\cos x + \sin x) = 1 - 2 \sin^2 x$

30. $(\sin x - \cos x)(\sin x + \cos x) = 1 - 2 \cos^2 x$

31. $\sin^2 x \cot^2 x + \sin^2 x = 1$

32. $\cos^2 x \tan^2 x + \cos^2 x = 1$

33. $\tan^2 x - \sin^2 x = \sin^4 x \sec^2 x$

34. $\cot^2 x - \cos^2 x = \cos^4 x \csc^2 x$

35. $\sin^3 x - \cos^3 x = (\sin x - \cos x)(1 + \sin x \cos x)$

36. $\sin^3 x + \cos^3 x = (\sin x + \cos x)(1 - \sin x \cos x)$

37. $\cos^4 x - \sin^4 x = 1 - 2 \sin^2 x$

38. $\cos^4 x - \sin^4 x = 2 \cos^2 x - 1$

39. $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2 \sec^2 x$

40. $\dfrac{1}{1 - \cos x} + \dfrac{1}{1 + \cos x} = 2 \csc^2 x$

41. $\dfrac{\sin x}{1 - \sin x} - \dfrac{\sin x}{1 + \sin x} = 2 \tan^2 x$

42. $\dfrac{\cos x}{1 - \cos x} - \dfrac{\cos x}{1 + \cos x} = 2 \cot^2 x$

43. $\dfrac{1}{\sec x - \tan x} + \dfrac{1}{\sec x + \tan x} = \dfrac{2}{\cos x}$

44. $\dfrac{1}{\csc x + \cot x} + \dfrac{1}{\csc x - \cot x} = \dfrac{2}{\sin x}$

45. $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$

46. $(\sin x - \cos x)^2 = 1 - 2 \sin x \cos x$

47. $(1 + \tan x)^2 = \sec^2 x + 2 \tan x$

48. $(1 - \cot x)^2 = \csc^2 x - 2 \cot x$

49. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$

50. $\cot^2 x + \tan^2 x = \sec^2 x \csc^2 x - 2$

Building Skills

In Exercises 9–28, find the exact value of each expression.

9. $\sin(45° + 30°)$ 10. $\sin(45° - 30°)$

11. $\sin(60° - 45°)$ 12. $\sin(60° + 45°)$

13. $\sin(-105°)$ 14. $\cos 285°$

15. $\tan 225°$ 16. $\tan(-165°)$

17. $\sin\left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)$ 18. $\cos\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$

19. $\tan\left(\dfrac{\pi}{4} - \dfrac{\pi}{6}\right)$ 20. $\cot\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right)$

21. $\sec\left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$ 22. $\csc\left(\dfrac{\pi}{4} - \dfrac{\pi}{3}\right)$

23. $\cos\dfrac{-5\pi}{12}$ 24. $\sin\dfrac{7\pi}{12}$

25. $\tan\dfrac{19\pi}{12}$ 26. $\sec\dfrac{\pi}{12}$

27. $\tan\dfrac{17\pi}{12}$ 28. $\csc\dfrac{11\pi}{12}$

In Exercises 29–42, verify each identity.

29. $\sin\left(x + \dfrac{\pi}{2}\right) = \cos x$ 30. $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$

31. $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos x$ 32. $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$

33. $\tan\left(x + \dfrac{\pi}{2}\right) = -\cot x$ 34. $\tan\left(x - \dfrac{\pi}{2}\right) = -\cot x$

35. $\csc(x + \pi) = -\csc x$ 36. $\sec(x + \pi) = -\sec x$

37. $\cos\left(x + \dfrac{3\pi}{2}\right) = \sin x$ 38. $\cos\left(x - \dfrac{3\pi}{2}\right) = -\sin x$

39. $\tan\left(x - \dfrac{3\pi}{2}\right) = -\cot x$ 40. $\tan\left(x + \dfrac{3\pi}{2}\right) = -\cot x$

41. $\cot(3\pi - x) = -\cot x$ 42. $\csc\left(\dfrac{5\pi}{2} - x\right) = \sec x$

In Exercises 43–52, find the exact value of each expression without using a calculator.

43. $\sin 56° \cos 34° + \cos 56° \sin 34°$

44. $\cos 57° \cos 33° - \sin 57° \sin 33°$

45. $\cos 331° \cos 61° + \sin 331° \sin 61°$

46. $\cos 110° \sin 70° + \sin 110° \cos 70°$

47. $\dfrac{\tan 129° - \tan 84°}{1 + \tan 129° \tan 84°}$

48. $\dfrac{\tan 28° + \tan 17°}{1 - \tan 28° \tan 17°}$

49. $\sin \dfrac{7\pi}{12} \cos \dfrac{3\pi}{12} - \cos \dfrac{7\pi}{12} \sin \dfrac{3\pi}{12}$

50. $\cos \dfrac{5\pi}{12} \cos \dfrac{\pi}{12} - \sin \dfrac{5\pi}{12} \sin \dfrac{\pi}{12}$

51. $\dfrac{\tan \dfrac{5\pi}{12} - \tan \dfrac{2\pi}{12}}{1 + \tan \dfrac{5\pi}{12} \tan \dfrac{2\pi}{12}}$

52. $\dfrac{\tan \dfrac{5\pi}{12} + \tan \dfrac{7\pi}{12}}{1 - \tan \dfrac{5\pi}{12} \tan \dfrac{7\pi}{12}}$

In Exercises 53–58, find the exact value of each expression, given that $\tan u = \dfrac{3}{4}$, **with** u **in quadrant III, and** $\sin v = \dfrac{5}{13}$, **with** v **in quadrant II.**

53. $\sin (u - v)$

54. $\sin (u + v)$

55. $\cos (u + v)$

56. $\cos (u - v)$

57. $\tan (u + v)$

58. $\tan (u - v)$

In Exercises 59–64, find the exact value of each expression, given that $\cos \alpha = -\dfrac{2}{5}$, **with** α **in quadrant II, and**

$\sin \beta = -\dfrac{3}{7}$, **with** β **in quadrant IV.**

59. $\sin (\alpha - \beta)$

60. $\cos (\alpha - \beta)$

61. $\csc (\alpha + \beta)$

62. $\sec (\alpha + \beta)$

63. $\cot (\alpha - \beta)$

64. $\cot (\alpha + \beta)$

Building Skills

In Exercises 9–14, use the given information about the angle θ to find the exact value of

a. $\sin 2\theta$ b. $\cos 2\theta$ c. $\tan 2\theta$

9. $\sin \theta = \dfrac{3}{5}$, θ in quadrant II

10. $\cos \theta = -\dfrac{5}{13}$, θ in quadrant III

11. $\tan \theta = 4$, $\sin \theta < 0$

12. $\sec \theta = -\sqrt{3}$, $\sin \theta > 0$

13. $\tan \theta = -2$, $\dfrac{\pi}{2} < \theta < \pi$

14. $\cot \theta = -7$, $\dfrac{3\pi}{2} < \theta < 2\pi$

In Exercises 15–24, use a double-angle formula to find the exact value of each expression.

15. $1 - 2\sin^2 75°$

16. $\dfrac{2\tan 75°}{1 - \tan^2 75°}$

17. $2\cos^2 105° - 1$

18. $1 - 2\sin^2 165°$

19. $\dfrac{2\tan 165°}{1 - \tan^2 165°}$

20. $2\cos^2 165° - 1$

21. $1 - 2\sin^2 \dfrac{\pi}{8}$

22. $2\cos^2\left(-\dfrac{\pi}{8}\right) - 1$

23. $\dfrac{2\tan\left(-\dfrac{5\pi}{12}\right)}{1 - \tan^2\left(-\dfrac{5\pi}{12}\right)}$

24. $1 - 2\sin^2\left(-\dfrac{7\pi}{12}\right)$

In Exercises 25 and 26, verify each "quadruple-angle" formula.

25. $\sin 4\theta = \cos\theta(4\sin\theta - 8\sin^3\theta)$

26. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

In Exercises 27–40, verify each identity.

27. $\cos^4 x - \sin^4 x = \cos 2x$

28. $1 + \cos 2x + 2\sin^2 x = 2$

29. $(\sin x - \cos x)^2 = 1 - \sin 2x$

30. $(\sin x + \cos x)^2 = 1 + \sin 2x$

31. $\sin 4x = 4\sin x \cos x \cos 2x$

32. $\sin 4x = 8\sin x \cos^3 x - 4\sin x \cos x$

33. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$

34. $\dfrac{\cos 3x}{\sin x} + \dfrac{\sin 3x}{\cos x} = 2\cot 2x$

35. $\dfrac{1 - \cos 2x}{\sin 2x} = \tan x$

36. $\dfrac{1 + \cos 2x}{\sin 2x} = \cot x$

37. $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$

38. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

39. $\dfrac{1 + \sin 2x}{\cos 2x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

40. $\cot x - \tan x = 2\cot 2x$

In Exercises 41–50, use the power-reducing formulas to rewrite each expression that does not contain trigonometric functions of power greater than 1.

41. $4 \sin^2 x \cos^2 x$

42. $\sin^2 x \cos^2 x$

43. $4 \sin x \cos x (1 - 2 \sin^2 x)$

44. $4 \sin x \cos x (2 \cos^2 x - 1)$

45. $2 \sin 3x \cos 3x (2 \cos^2 3x - 1)$

46. $\sin 8x (1 - 2 \sin^2 4x)$

47. $\sin \dfrac{x}{2} \cos \dfrac{x}{2} \left(1 - 2 \sin^2 \dfrac{x}{2} \right)$

48. $\sin x \left(2 \cos^2 \dfrac{x}{2} - 1 \right)$

49. $8 \sin^4 \dfrac{x}{2}$

50. $8 \cos^4 \dfrac{x}{2}$

In Exercises 51–62, use half-angle formulas to find the exact value of each expression.

51. $\sin \dfrac{\pi}{12}$

52. $\sin \dfrac{\pi}{8}$

53. $\cos \dfrac{\pi}{8}$

54. $\tan \dfrac{\pi}{8}$

55. $\sin \left(-\dfrac{3\pi}{8} \right)$

56. $\cos \left(-\dfrac{3\pi}{8} \right)$

57. $\tan \left(\dfrac{7\pi}{8} \right)$

58. $\sec \left(-\dfrac{7\pi}{8} \right)$

59. $\tan 112.5°$

60. $\cos 112.5°$

61. $\sin (-75°)$

62. $\tan (-105°)$

Building Skills

In Exercises 9–24, use the product-to-sum formulas to rewrite each expression as the sum or difference of two functions. Simplify where possible.

9. $\sin x \cos x$

10. $\cos x \cos x$

11. $\sin x \sin x$

12. $\cos x \sin x$

13. $\sin 25° \cos 5°$

14. $\sin 40° \sin 20°$

15. $\cos 140° \cos 20°$

16. $\cos 70° \sin 20°$

17. $\sin \dfrac{7\pi}{12} \sin \dfrac{\pi}{12}$

18. $\sin \dfrac{3\pi}{8} \cos \dfrac{\pi}{8}$

19. $\cos \dfrac{5\pi}{8} \sin \dfrac{\pi}{8}$

20. $\cos \dfrac{5\pi}{3} \cos \dfrac{\pi}{3}$

21. $\sin 5\theta \cos \theta$

22. $\cos 3\theta \sin 2\theta$

23. $\cos 4x \cos 3x$

24. $\sin 5x \sin 2x$

In Exercises 25–32, find the exact value of each expression.

25. $\sin 37.5° \sin 7.5°$

26. $\cos 52.5° \cos 7.5°$

27. $\sin 67.5° \cos 22.5°$

28. $\cos 105° \sin 75°$

29. $\sin \dfrac{5\pi}{24} \cos \dfrac{\pi}{24}$

30. $\sin \dfrac{7\pi}{12} \sin \dfrac{\pi}{12}$

31. $\cos \dfrac{13\pi}{24} \cos \dfrac{5\pi}{24}$

32. $\cos \dfrac{7\pi}{24} \sin \dfrac{\pi}{24}$

In Exercises 33–52, use sum-to-product formulas to rewrite each expression as a product. Simplify where possible.

33. $\cos 40° - \cos 20°$

34. $\sin 22° + \sin 8°$

35. $\sin 32° - \sin 16°$

36. $\cos 47° + \cos 13°$

37. $\sin \dfrac{\pi}{5} + \sin \dfrac{2\pi}{5}$

38. $\cos \dfrac{\pi}{12} + \cos \dfrac{\pi}{3}$

39. $\cos \dfrac{1}{2} + \cos \dfrac{1}{3}$

40. $\sin \dfrac{2}{3} - \sin \dfrac{1}{4}$

41. $\cos 3x + \cos 5x$

42. $\sin 5x - \sin 3x$

43. $\sin 7x + \sin (-x)$

44. $\cos 7x - \cos 3x$

45. $\sin x + \cos x$

46. $\cos x - \sin x$

47. $\sin 2x - \cos 2x$

48. $\cos 3x + \sin 3x$

49. $\sin 3x + \cos 5x$

50. $\sin 5x - \cos x$

51. $a(\sin x + \cos x)$

52. $a(\sin bx + \cos bx)$

In Exercises 53–62, verify each identity.

53. $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

54. $\dfrac{\sin 2x + \sin 4x}{\cos 2x + \cos 4x} = \tan 3x$

55. $\dfrac{\cos 3x - \cos 7x}{\sin 7x + \sin 3x} = \tan 2x$

56. $\dfrac{\cos 12x - \cos 4x}{\sin 4x - \sin 12x} = \tan 8x$

57. $\dfrac{\cos 2x + \cos 2y}{\cos 2x - \cos 2y} = \cot(y - x)\cot(y + x)$

58. $\dfrac{\sin 2x + \sin 2y}{\sin 2x - \sin 2y} = \dfrac{\tan(x + y)}{\tan(x - y)}$

59. $\sin x + \sin 2x + \sin 3x = \sin 2x(1 + 2\cos x)$

60. $\cos x + \cos 2x + \cos 3x = \cos 2x(1 + 2\cos x)$

61. $\sin 2x + \sin 4x + \sin 6x = 4\cos x \cos 2x \sin 3x$

62. $\cos x + \cos 3x + \cos 5x + \cos 7x = 4\cos x \cos 2x \cos 4x$

Building Skills

In Exercises 9–18, find all solutions of each equation. Express the solutions in radians.

9. $\cos x = 0$

10. $\sin x = 0$

11. $\tan x = -1$

12. $\cot x = -1$

13. $\cos x = \dfrac{\sqrt{2}}{2}$

14. $\sin x = \dfrac{\sqrt{3}}{2}$

15. $\cot x = \sqrt{3}$

16. $\tan x = -\dfrac{\sqrt{3}}{3}$

17. $\cos x = -\dfrac{1}{2}$

18. $\sin x = -\dfrac{\sqrt{3}}{2}$

In Exercises 19–28, find all solutions of each equation. Express the solutions in degrees.

19. $\tan x = \dfrac{\sqrt{3}}{3}$

20. $\cot x = 1$

21. $\sin x = -\dfrac{1}{2}$

22. $\cos x = \dfrac{1}{2}$

23. $\csc x = 1$

24. $\sec x = -1$

25. $\sqrt{3}\csc x - 2 = 0$

26. $\sqrt{3}\sec x + 2 = 0$

27. $2\sec x - 4 = 0$

28. $2\csc x + 4 = 0$

In Exercises 35–40, find all solutions of each equation in the interval $[0, 2\pi)$. Round the solutions to four decimal places.

35. $\csc x = -2$

36. $3\sin x - 1 = 0$

37. $3\tan x + 4 = 0$

38. $2\sec x - 7 = 0$

39. $2\csc x + 5 = 0$

40. $\cos x = 0.1106$

In Exercises 41–48, find all solutions of each equation in the interval $[0, 2\pi)$.

41. $\sin\left(x + \dfrac{\pi}{4}\right) = \dfrac{1}{2}$

42. $2\cos\left(x - \dfrac{\pi}{4}\right) + 1 = 0$

43. $\sec\left(x - \dfrac{\pi}{8}\right) + 2 = 0$

44. $\csc\left(x + \dfrac{\pi}{8}\right) - 2 = 0$

45. $\sqrt{3}\tan\left(x - \dfrac{\pi}{6}\right) - 1 = 0$

46. $\cot\left(x + \dfrac{\pi}{6}\right) + 1 = 0$

47. $2\sin\left(x - \dfrac{\pi}{3}\right) + 1 = 0$

48. $2\cos\left(x + \dfrac{\pi}{3}\right) + \sqrt{2} = 0$

In Exercises 49–68, find all solutions of each equation in the interval $[0, 2\pi)$.

49. $\cos 2x = \dfrac{\sqrt{3}}{2}$

50. $\sin 2x = \dfrac{\sqrt{3}}{2}$

51. $\sin 2x = -\dfrac{1}{2}$

52. $\cos 2x = -\dfrac{1}{2}$

53. $\sec 2x = \dfrac{1}{2}$ **54.** $\csc 2x = \dfrac{\sqrt{3}}{2}$

55. $\tan 2x = \dfrac{\sqrt{3}}{3}$

56. $\cot 2x = \dfrac{\sqrt{3}}{3}$

57. $\sin\left(2x - \dfrac{\pi}{3}\right) = \dfrac{1}{2}$

58. $\cos\left(2x + \dfrac{\pi}{4}\right) = -\dfrac{\sqrt{3}}{2}$

59. $\sin 3x = \dfrac{1}{2}$

60. $\cos 3x = -\dfrac{1}{2}$

61. $\cos 3x = \dfrac{1}{2}$

62. $\sin 3x = -\dfrac{\sqrt{3}}{2}$

63. $\cos \dfrac{x}{2} = \dfrac{1}{2}$ **64.** $\csc \dfrac{x}{2} = 2$

65. $\sin \dfrac{x}{2} = -\dfrac{\sqrt{3}}{2}$ **66.** $\sec \dfrac{x}{2} = -\dfrac{2\sqrt{3}}{3}$

67. $\tan \dfrac{x}{3} = 1$ **68.** $\cot \dfrac{x}{3} = \sqrt{3}$

In Exercises 69–78, find all solutions of each equation in the interval $[0, 2\pi)$.

69. $\sin 2x = \sin x$

70. $\cos 2x = \cos x$

71. $\sin 2x = \cos x$

72. $\cos 2x = \sin x$

73. $\cos\left(2x + \dfrac{\pi}{4}\right) = \cos x$

74. $\sin\left(2x - \dfrac{\pi}{4}\right) = \sin x$

75. $\sin 2x = \cos\left(x - \dfrac{\pi}{4}\right)$

76. $\cos\left(2x - \dfrac{\pi}{4}\right) = \sin x$

77. $\sin\left(2x - \dfrac{\pi}{4}\right) = \cos\left(x + \dfrac{\pi}{4}\right)$

78. $\cos\left(2x + \dfrac{\pi}{4}\right) = \sin\left(x - \dfrac{\pi}{4}\right)$

In Exercises 79–86, find all solutions of each equation in the interval $[0, 2\pi)$.

79. $(\sin x + 1)(\tan x - 1) = 0$

80. $(2\cos x + 1)(\sqrt{3}\tan x - 1) = 0$

81. $(\csc x - 2)(\cot x + 1) = 0$

82. $(\sqrt{3}\sec x - 2)(\sqrt{3}\cot x + 1) = 0$

83. $(\tan x + 1)(2\sin x - 1) = 0$

84. $(2\sin x - \sqrt{3})(2\cos x - 1) = 0$

85. $(\sqrt{2}\sec x - 2)(2\sin x + 1) = 0$

86. $(\cot x - 1)(\sqrt{2}\csc x + 2) = 0$

In Exercises 87–102, find all solutions of each equation in the interval $[0, 2\pi)$.

87. $4\sin^2 x = 1$　　　　　　**88.** $4\cos^2 x = 1$

89. $\tan^2 x = 1$　　　　　　**90.** $\sec^2 x = 2$

91. $3\csc^2 x = 4$　　　　　　**92.** $3\cot^2 x = 1$

93. $3\sin^2 x = \cos^2 x$　　　　**94.** $3\cos^2 x = \sin^2 x$

95. $\cos^2 x - \sin^2 x = 1$

96. $2\sin^2 x + \cos x - 1 = 0$

97. $2\cos^2 x - 3\sin x - 3 = 0$

98. $2\sin^2 x - \cos x - 1 = 0$

99. $2\sin^2\theta - \sin\theta - 1 = 0$

100. $2\cos^2\theta - 5\cos\theta + 2 = 0$

101. $\sqrt{3}\sec^2 x - 2\tan x - 2\sqrt{3} = 0$

102. $\csc^2 x - (\sqrt{3} + 1)\cot x + (\sqrt{3} - 1) = 0$

In Exercises 103–108, solve each trigonometric equation in the interval $[0, 2\pi)$ by squaring both sides.

103. $\sqrt{3}\sin x = 1 + \cos x$

104. $1 + \sin x = \sqrt{3}\cos x$

105. $\tan x + 1 = \sec x$

106. $\sqrt{3}\tan\theta + 1 = \sqrt{3}\sec\theta$

107. $\sqrt{3}\cot\theta + 1 = \sqrt{3}\csc\theta$

108. $\sin x + \cos x = \sqrt{1 - \sin 2x}$