

Name: Key

GTID: _____

Answer the questions in the spaces provided and put your answer in the **BOX**. Organize work clearly and simplify answers for full credit.

- 1a. Find the exact values: $\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$ and $\sec\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

Formula:

$$\cos(\theta + \alpha) = \cos \theta \cos \alpha - \sin \theta \sin \alpha$$

$$\begin{aligned}\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{\sqrt{2} - \sqrt{6}}{4} \\ \sec \theta &= -\sqrt{2} - \sqrt{6}\end{aligned}$$

$$\begin{array}{c} \text{Diagram 1: } \begin{array}{c} \text{Right triangle with hypotenuse } \sqrt{2}, \text{ angle } \frac{\pi}{3} \text{ at the bottom left, and angle } \frac{\pi}{4} \text{ at the top right.} \\ \text{Angle } u = \frac{\pi}{3}, \text{ angle } v = \frac{\pi}{4}. \end{array} \\ \begin{array}{c} \text{Diagram 2: } \begin{array}{c} \text{Right triangle with hypotenuse } 1, \text{ angle } \frac{\pi}{3} \text{ at the bottom left, and angle } \frac{\pi}{4} \text{ at the top right.} \\ \text{Angle } u = \frac{\pi}{3}, \text{ angle } v = \frac{\pi}{4}. \end{array} \\ \begin{array}{c} \text{Equation: } \frac{\sqrt{2} - \sqrt{6}}{4} \\ \boxed{\frac{\sqrt{2} - \sqrt{6}}{4}} \end{array} \end{array} \end{array}$$

$$\sec\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{4}{\sqrt{2} - \sqrt{6}} \cdot \frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} + \sqrt{6}} = \frac{4\sqrt{2} + 4\sqrt{6}}{2 - 6} = \frac{4\sqrt{2} + 4\sqrt{6}}{-4} = \boxed{-\sqrt{2} - \sqrt{6}}$$

- 1b. Given that $\tan u = \frac{3}{4}$, with u in quadrant III, and $\sin v = \frac{5}{13}$, with v in quadrant II, find the exact value of $\sin(u+v)$.

$$\begin{array}{c} \text{Diagram 1: } \begin{array}{c} \text{Right triangle with hypotenuse } 5, \text{ angle } u \text{ at the top left, and angle } v \text{ at the top right.} \\ \text{Angle } u = \frac{3}{5}, \text{ angle } v = \frac{4}{5}. \end{array} \\ \begin{array}{c} \text{Diagram 2: } \begin{array}{c} \text{Right triangle with hypotenuse } 5, \text{ angle } u \text{ at the top left, and angle } v \text{ at the top right.} \\ \text{Angle } u = \frac{3}{5}, \text{ angle } v = \frac{4}{5}. \end{array} \end{array} \end{array}$$

$$\begin{array}{c} \text{Formula:} \\ \sin(u+v) = \sin u \cos v + \cos u \sin v \end{array}$$

$$\boxed{\frac{16}{65}}$$

$$\begin{aligned}&= \left(-\frac{3}{5}\right) \left(\frac{-12}{13}\right) + \left(\frac{4}{5}\right) \left(\frac{5}{13}\right) \\ &= \frac{36}{65} - \frac{20}{65} = \boxed{\frac{16}{65}}\end{aligned}$$

1c. Find the exact value: $2 \sin\left(-\frac{\pi}{8}\right) \cos\left(-\frac{\pi}{8}\right)$

Formula: $\sin 2\theta = 2 \sin \theta \cos \theta$

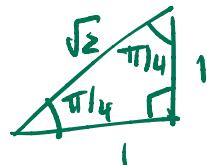
$$2 \sin\left(-\frac{\pi}{8}\right) \cos\left(-\frac{\pi}{8}\right) = \sin\left(2 \cdot -\frac{\pi}{8}\right)$$

$$-\frac{\sqrt{2}}{2}$$

$$= \sin\left(-\frac{\pi}{4}\right)$$

$$= -\sin(\pi/4) \quad \text{b/c } y = \sin x \text{ is ODD}$$

$$= \boxed{-\frac{\sqrt{2}}{2}}$$



1d. Use the half angle identity to find: $\sin(22.5^\circ)$

Formula:

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

$$\sin(22.5^\circ) = \sqrt{\frac{1 - \cos(45^\circ)}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2 - \sqrt{2}}{2} \cdot \frac{1}{2}}$$

$$= \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

1e. Simplify: $(1 + \tan \theta)(1 - \tan \theta) + \sec^2 \theta$

$$(1 + \tan \theta)(1 - \tan \theta) + \sec^2 \theta \\ = 1 - \tan^2 \theta + \sec^2 \theta$$

$$= 1 - \cancel{\tan^2 \theta} + \cancel{\tan^2 \theta} + 1$$

$$= \boxed{2}$$

2

Formula:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\Rightarrow \tan^2 \theta + 1 = \sec^2 \theta$$

1f. Find all solutions in the interval $[0, 2\pi]$ of: $\sin 2x = -\frac{\sqrt{3}}{2}$

Set $\theta = 2x$

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

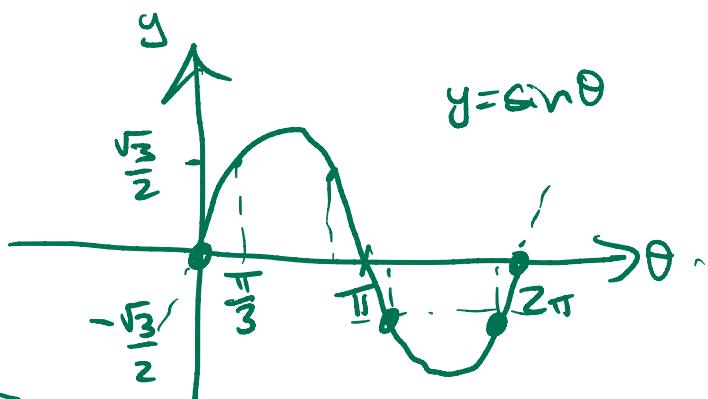
$$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

also next loop

$$\theta = \frac{4\pi}{3} + 2\pi = \frac{10\pi}{3}$$

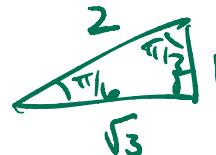
$$\text{so } x = \frac{\theta}{2}$$

$$\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$$



$$\theta = \frac{5\pi}{3} + 2\pi = \frac{11\pi}{3}$$

$$x = \frac{4\pi}{6}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{3}$$



2. Use power-reducing formulas to rewrite the expression so that it does not contain trigonometric functions of power greater than 1.

$$\cos^4 x$$

Formula: $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$

$$\begin{aligned} \cos^4 \theta &= (\cos^2 \theta)^2 = \left(\frac{1+\cos 2\theta}{2} \right)^2 = \frac{1+2\cos 2\theta + \cos^2 2\theta}{4} \\ &= \frac{1}{4} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \right) = \frac{1}{4} \left(1 + 2\cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta \right) \\ &= \boxed{\frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta} \end{aligned}$$

3. Verify the identity.

$$\begin{aligned} &\tan^2 x - \sin^2 x \\ &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x \\ &= \frac{\sin^2 x - \sin^2 x \cos^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x (1 - \cos^2 x)}{\cos^2 x} \end{aligned}$$

$\tan^2 x - \sin^2 x = \sin^4 x \sec^2 x$
 \downarrow
 $= \frac{\sin^2 x \cdot \overbrace{\sin^2 x}^{1 - \cos^2 x}}{\cos^2 x}$
 $= \sin^4 x \cdot \frac{1}{\cos^2 x}$
 $= \sin^4 x \sec^2 x \quad \checkmark$

Formula:
 $\sin^2 x + \cos^2 x = 1$
 $\Rightarrow 1 - \cos^2 x = \sin^2 x$

4. Solve the equation for all values in the range $[-2\pi, 2\pi]$.

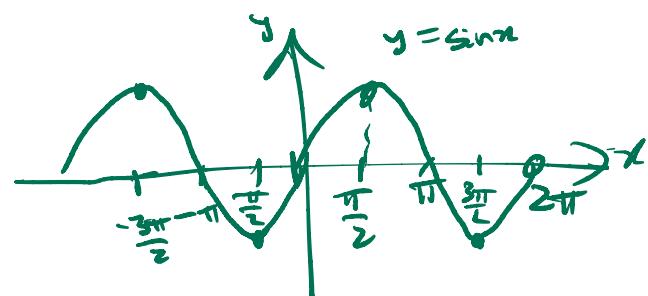
$$(1 - \sin x)(1 + \sin x) = 0$$

$$1 - \sin x = 0 \Rightarrow \sin x = 1$$

$$\Rightarrow x = \frac{\pi}{2} + 2\pi n, n \text{ integer}$$

$$1 + \sin x = 0 \Rightarrow \sin x = -1$$

$$\Rightarrow x = \frac{3\pi}{2} + 2\pi n, n \text{ integer.}$$



$$x = \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$$

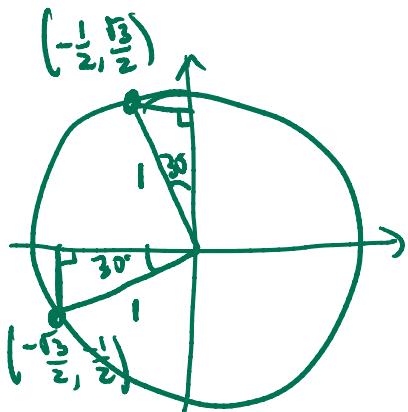
5. Find the exact value: $\sin(165^\circ) \cdot \cos(45^\circ)$

Hint: use the product-to-sum formula.

Formula:

$$\sin \theta \cos \alpha = \frac{1}{2} [\sin(\theta+\alpha) + \sin(\theta-\alpha)]$$

$$\sin(165^\circ) \cos(45^\circ) = \frac{1}{2} [\sin(165^\circ + 45^\circ) + \sin(165^\circ - 45^\circ)]$$



$$= \frac{1}{2} [\sin(210^\circ) + \sin(120^\circ)]$$

$$= \frac{1}{2} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} \right) = \boxed{\frac{-1 + \sqrt{3}}{4}}$$