Practice Exam 3

1. Show that $f(x) = \frac{3}{4x^2+2}$ satisfies $f(x) = O(1)$.

Need to show that $\lim_{x \to \infty} \frac{f(x)}{1} \leq M$, for some number $M > 0$.

$\lim_{x \to \infty} \frac{f(x)}{1} = \lim_{x \to \infty} \frac{3}{4x^2+2} = \underline{0}$, which is ≤ 1, for example.

2. Find the particular solution to the separable differential equation $y' = \sqrt{\frac{x}{y}}$ satisfying $f(0) = \sqrt{3}$.

$\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{y}} \Rightarrow \int \sqrt{y}\, dy = \int \sqrt{x}\, dx$

$\Rightarrow \frac{2}{3} y^{3/2} = \frac{2}{3} x^{3/2} + C.$

$\frac{2}{3}(\sqrt{3})^{3/2} = \frac{2}{3}(0)^{3/2} + C.$

$C = \frac{2 \cdot 3^{3/4}}{3} = \frac{2}{\sqrt[4]{3}}.$

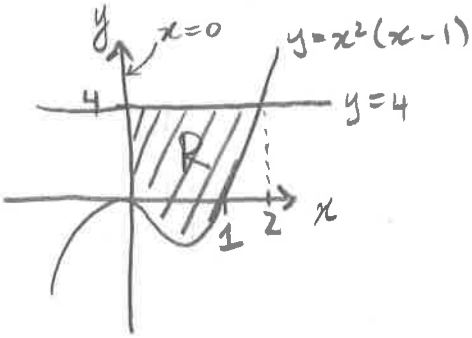
implicit answer

$\frac{2}{3} y^{3/2} = \frac{2}{3} x^{3/2} + \frac{2}{\sqrt[4]{3}}$

explicit answer: $\boxed{y = \left(x^{3/2} + 3^{3/4}\right)^{2/3}}$

3. Let R be the region bounded by the curves $y = x^2(x-1)$, $y = 4$ and $x = 0$.

(i) Find the area of R.



$$\text{Area}(R) = \int_0^2 (\text{TOP} - \text{BOT})\, dx$$

$$= \int_0^2 \left(4 - x^2(x-1)\right) dx = \ldots =$$

$$= \boxed{20/3}$$

(ii) Find the x-value of the center of mass of R.

$\bar{x} = \dfrac{M_y}{M}$ where $M_y = \int \tilde{x}\, dm$ and $M = \int dm$

(where $\tilde{x} = x$ and $dm = (\text{TOP}-\text{BOT})\, dx = (4 - x^2(x-1))\, dx$

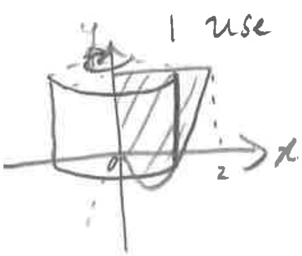
$M = \text{Area}(R) = 20/3$

$M_y = \int_0^2 x\left(4 - x^2(x-1)\right) dx = \ldots = 28/5$

So $\bar{X} = \dfrac{28/5}{20/3} = \dfrac{28 \cdot 3}{5 \cdot 20} = \dfrac{7 \cdot 3}{5 \cdot 5} = \boxed{\dfrac{21}{25}}$

(iii) Find the volume of the solid obtained by rotating R about the y-axis.

I use cylindrical shell method:



$$V = \int_0^2 2\pi x \cdot \left(4 - x^2(x-1)\right) dx$$

$= 2\pi \cdot M_y$ ((coincidence?)

$= 2\pi \cdot 28/5 = \boxed{56\pi/5}$

4. Consider the curve $y = x^{3/2}$ between $x = 0$ to $x = 2$. Find the arc length of the curve.

$$\text{arc length} = \int_0^2 \sqrt{1 + [(x^{3/2})']}\ dx$$

$$= \int_0^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\ dx = \int_0^2 \sqrt{1 + \frac{9}{4}x}\ dx$$

u-sub

$u = 1 + \frac{9}{4}x$

$du = \frac{9}{4}\,dx$

$$= \frac{4}{9} \cdot \frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2} \Big|_0^2$$

$$= \frac{8}{27}\left[\left(1 + \frac{9}{4}(2)\right)^{3/2} - 1\right] = \frac{8}{27}\left[\left(\frac{11}{2}\right)^{3/2} - 1\right] \approx \boxed{3.525}$$

5. Find the surface area of the surface obtained by rotating the curve $y = x^3$, $0 \leq x \leq 2$, about the x-axis.

$$\int_0^2 2\pi x^3 \sqrt{1 + [(x^3)']^2}\ dx = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\ dx$$

u-sub

$u = 1 + 9x^4$

$du = 36x^3\,dx$

$$= 2\pi \frac{(1 + 9x^4)^{3/2}}{54} \Big|_0^2$$

$$\approx \boxed{32.315 \cdot 2\pi}$$

$$= \boxed{203.0411}$$

6. Show that $y = \int_1^x \frac{1}{t}\, dt - 2$ satisfies the initial value problem $\frac{dy}{dx} = \frac{1}{x}$ and $y(1) = -2$.

$$\underline{\frac{dy}{dx} = \frac{1}{x}} \quad \text{and} \quad \underline{y(1)} = \int_1^1 \frac{1}{t}\, dt - 2$$

$$= 0 - 2 = \underline{-2} \checkmark$$

7. Consider the sequences $a_n = \frac{n^2+1}{n+1}$ and $b_n = \ln(n) - \ln(n+1)$. Do the sequences a_n and b_n converge or diverge?

a_n <u>diverges</u> to ∞.

$$b_n = \ln(n) - \ln(n+1)$$

$$= \ln\left(\frac{n}{n+1}\right)$$ <u>converges</u> to $0 = \ln(1)$

since

$\frac{n}{n+1}$ converges to 1

as $n \to \infty$.

8. Suppose the force to move an object over 5 feet is given by $F(x) = \frac{x^2}{\sqrt{30-x^2}}$, $0 \le x \le 5$, where x is the distance from start. Find the work needed to move the object 5 feet.

trig-sub

$$\boxed{\begin{array}{l} x = \sqrt{30}\sin\theta \\ dx = \sqrt{30}\cos\theta\, d\theta \end{array}}$$

$$W = \int_0^5 \frac{x^2}{\sqrt{30-x^2}}\, dx$$

$$= \int_*^* \frac{30\sin^2\theta}{\sqrt{30}\,|\cos\theta|}\sqrt{30}\cos\theta\, d\theta = \int_*^* 30\sin^2\theta\, d\theta$$

$$= 15\int_*^* (1-\cos 2\theta)\, d\theta = 15\theta - \frac{15}{2}(\underbrace{\sin 2\theta}_{(2\sin\theta\cos\theta)})\Big|_*^*$$

$$= 15\sin^{-1}\left(\frac{x}{\sqrt{30}}\right) - \frac{15x}{\sqrt{30}}\cdot\frac{\sqrt{30-x^2}}{\sqrt{30}}\Big|_0^5 = \boxed{15\sin^{-1}\left(\frac{5}{\sqrt{30}}\right) - \frac{5\sqrt{5}}{2}}$$

9. Integrate $\int \tan^4 x\, dx$.

$$= \int \tan^2 x \cdot \tan^2 x\, dx$$

$$= \int (\sec^2 x - 1)\tan^2 x\, dx$$

$$= \int (\tan^2 x\sec^2 x - \tan^2 x)\, dx$$

$$= \int \tan^2 x\sec^2 x\, dx - \int \tan^2 x\, dx$$

$$= \int u^2\, du - \int (\sec^2 x - 1)\, dx$$

$$= \frac{1}{3}u^3 - \int \sec^2 x\, dx + \int dx$$

$$= \boxed{\frac{1}{3}\tan^3 x - \tan x + x + C}$$