Exam 3 Answers

1. Integrate $\displaystyle\int \frac{\ln(x^{2/3})}{x}\,dx.$ (10 pts.)

u-sub

$$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$$

$$= \frac{2}{3}\int \frac{\ln(x)}{x}\,dx$$

$$= \frac{2}{3}\int u\,du$$

$$= \frac{2}{3}\cdot\frac{1}{2}u^2 + C$$

$$= \frac{1}{3}[\ln(x)]^2 + C$$

$$\boxed{\frac{1}{3}[\ln(x)]^2 + C}$$

2. Find the particular solution $y = f(x)$ which satisfies the separable differential equation $2y'\sqrt{x} = e^{\sqrt{x}-y}$ and $y(1) = 0$. (10 pts.)

$$2\,\frac{dy}{dx}\sqrt{x} = \frac{e^{\sqrt{x}}}{e^y}$$

$$\int e^y\,dy = \int \frac{1}{2\sqrt{x}}e^{\sqrt{x}}\,dx$$

$$e^y = e^{\sqrt{x}} + C$$

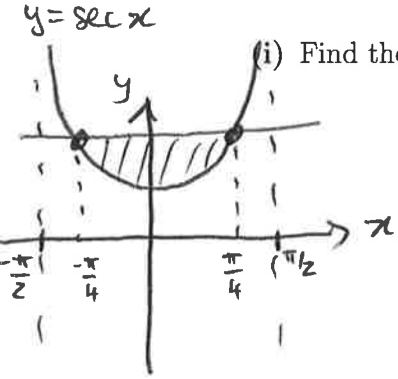
$$y = \ln(e^{\sqrt{x}} + C)$$

$$0 = \ln(e + C)$$

$$C = -e + 1$$

$$\boxed{y = \ln(e^{\sqrt{x}} - e + 1)}$$

3. Let R be the region bounded by one arc of the secant curve $y = \sec x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and the line $y = \sqrt{2}$.



(i) Find the area of R. (10 pts.)

$$\text{Area}(R) = \int_{-\pi/4}^{\pi/4} (\sqrt{2} - \sec x)\, dx$$

$$= 2\int_0^{\pi/4} (\sqrt{2} - \sec x)\, dx$$

$$= 2\left(\sqrt{2}\,x - \ln|\sec x + \tan x| \Big|_0^{\pi/4}\right)$$

$$= 2\left(\sqrt{2}\cdot \pi/4 - \ln|\sqrt{2}+1| - (0-0)\right)$$

$$\boxed{\frac{\sqrt{2}\pi}{2} - \ln(\sqrt{2}+1) \approx .46}$$

(ii) Find the volume of the solid obtained by rotating R about the x-axis. (10 pts)

$$\text{Volume} = \int_{-\pi/4}^{\pi/4} \pi\left[(\sqrt{2})^2 - (\sec x)^2\right] dx$$

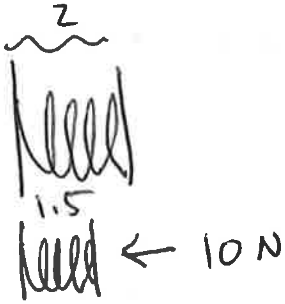
$$= 2\pi \int_0^{\pi/4} (2 - \sec^2 x)\, dx$$

$$= 2\pi\left(2x - \tan x \Big|_0^{\pi/4}\right)$$

$$= 2\pi\left(2(\pi/4) - 1 - (0-0)\right) = \pi^2 - 2\pi$$

$$\boxed{\pi^2 - 2\pi \approx 3.58}$$

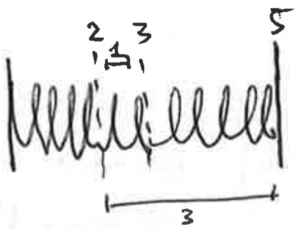
4. A certain spring is 2 meters long at rest, and it takes 10 N of force to compress the spring to 1.5 meters. Find the work needed to stretch the spring from 3 meters to 5 meters. (10 pts.)



$F = kx$

$10 = k * .5$

$\underline{k = 20}$



$W = \int_1^3 20x\,dx = 10x^2 \Big|_1^3 = 10(9-1) = 80$

$\boxed{80 \text{ joules}}$

5. A certain population of bacteria triples every 21 days. How long does it take for the number of bacteria to double? (10 pts.)

$P = P_0\, 3^{t/21}$

$2P_0 = P_0\, 3^{t/21}$ for what t?

$2 = 3^{t/21}$

$\ln(2) = \frac{t}{21} \ln(3)$

$t = \frac{21 \ln(2)}{\ln(3)}$

OR

$\frac{dP}{dt} = kP \Rightarrow \int \frac{1}{P}\,dP = \int k\,dt$

$\Rightarrow \ln|P| = kt + C$

$\rightarrow P = P_0 e^{kt}$

& $3P_0 = P_0 e^{k \cdot 21}$

$\Rightarrow 3 = e^{k \cdot 21} \Rightarrow k = \ln(3)/21$

$P = P_0 e^{\left(\frac{\ln 3}{21}\right)t} = P_0\, 3^{t/21}$

(Then proceed as over ← there)

$\boxed{t = \frac{21 \ln(2)}{\ln(3)} = 21 \log_3(2)}$

6. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$. (10 pts.)

arc length $= \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$= \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos 2x}\right)^2}\, dx$

$= \int_0^{\pi/4} \sqrt{1 + \cos 2x}\, dx$

$= \int_0^{\pi/4} \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos x\, dx$

$= \sqrt{2} \sin x \Big|_0^{\pi/4}$

$= \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1.$

$$\boxed{1}$$

7. Find the value of the definite integral $\int_0^{\pi/4} x^2 \sin(2x)\, dx$. (10 pts.)

$\int_0^{\pi/4} x^2 \sin 2x = \frac{-x^2}{2} \cos 2x \Big|_0^{\pi/4} - \int_0^{\pi/4} 2x \cdot \frac{-1}{2} \cos 2x\, dx$

IBP

$u = x^2 \quad dv = \sin 2x$

$du = 2x\, dx \quad v = -\frac{1}{2}\cos 2x$

$= 0 + \int_0^{\pi/4} x \cos 2x\, dx$

$= \frac{x}{2} \sin 2x \Big|_0^{\pi/4} - \int_0^{\pi/4} \frac{1}{2} \sin 2x\, dx$

$= \frac{\pi}{8} - 0 - \frac{1}{2} \cdot \frac{1}{2} \cos(2x) \Big|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}(0 - 1)$

IBP

$u = x \quad dv = \cos 2x$

$du = dx \quad v = \frac{1}{2} \sin 2x$

$= \frac{\pi}{8} - \frac{1}{4}$

$$\boxed{\frac{\pi}{8} - \frac{1}{4}}$$

8. Which of the following statements are always true for functions f, g from $\mathbb{R}$ to $\mathbb{R}$? (Assume that f, g are eventually positive) (10 pts.)

(I) If f is a polynomial of degree n and g is a polynomial of degree m, and $n < m$, then $f = O(g)$.

(II) If $f = O(g)$ and $g = O(f)$, then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.

(III) Suppose $f = O(x)$ and $g = O(x^2)$, then $f = O(g)$.

(A) I only.
(B) II only.
(C) III only.
(D) I & III.
(E) None are true.

9. Which of the following sequences converge? (10 pts.)

(I) $a_n = \dfrac{(-1)^n}{n!}$,

(II) $b_n = \cos\left(\dfrac{n\pi}{n+1}\right)$,

(III) $c_n = \dfrac{e^n}{n^2}$.

(A) I only.
(B) II only.
(C) I & II.
(D) II & III.
(E) I, II, & III.