Worksheet 3: Chapter 2 (Limits and Continuity)

1. Find the domain of the function $f(x)$ with unknown constant parameter a

$$f(x) = \begin{cases} x^3 - x + 1 & \text{if } x \leq 5/4 \\ 1/2^x & \text{if } 5/4 < x \leq 2 \\ ax + 3 & \text{if } x > 2 \end{cases}$$

(a) Find the limits

$\lim_{x\to\frac{5}{4}^+} f(x) =$ $1/2^{5/4} \approx .42045$

$\lim_{x\to\frac{5}{4}} f(x) =$ DNE $\left\{ \text{b/c} \lim_{x\to 5/4^-} f(x) = \left(\frac{5}{4}\right)^3 - \frac{5}{4} + 1 = \frac{5^3}{64} - \frac{4^2 \cdot 5}{64} + \frac{64}{64} \right.$ why?

$\overset{!!}{\left[\lim_{x\to 5/4^-} f(x) \neq \lim_{x\to\frac{5}{4}^+} f(x)\right]}$ $= \frac{125 - 80 + 64}{64}$

$\lim_{x\to 2^-} f(x) =$ $1/2^2 = 1/4$

$= \frac{109}{64} \approx 1.703$

$\lim_{x\to 2} f(x) =$? Depends on a, since $\lim_{x\to 2^+} = 2a+3$ may not equal $\frac{1}{4}$.

(b) Is the function continuous at $x = 5/4$? Explain.

No. $\lim_{x\to 5/4} f(x)$ DNE so the function $f(x)$ can not be continous at $x = 5/4$.

(c) Find the value of a for which the function is continuous at $x = 2$.

We require

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x).$$

That means

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1/2^x = 1/4$

must equal

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax + 3 = 2a + 3$

So $\frac{1}{4} = 2a + 3$

$\Leftrightarrow \frac{1}{4} - 3 = 2a$

$-11/4 = 2a$

$a = -11/8$

2. Find any horizontal, vertical or oblique asymptotes for the following functions.

Note: $f(x)=0$ when $x=-1$ and is undefined at $x=-3,-1$.

(a) $f(x) = \frac{2x^2-2}{x^2+4x+3} = \frac{2(x^2-1)}{(x+3)(x+1)} = \frac{2(x+1)(x-1)}{(x+3)(x+1)}$

vertical asymptotes

$\boxed{\text{VA: at } x=-3. \quad \text{HA: at } y=2.}$

b/c. $\lim_{x\to -3^-} f(x) = +\infty$ DNE and $\lim_{x\to -3^+} f(x) = -\infty$ DNE

horizontal asymptote

(b) $g(x) = e^x$

$\boxed{\text{VA: none} \quad \text{HA: at } y=0}$ since $\lim_{x\to -\infty} e^x = 0$.

(c) $h(x) = \frac{3x^2-2x+1}{x-1} = \frac{3x^2-3x+x+1}{x-1} = \frac{3x(x-1)+x+1}{x-1}$

$= 3x + \frac{x+1}{x-1} = 3x + \frac{x-1+1+1}{x-1} = \underbrace{3x+1}_{\text{linear part}} + \underbrace{\frac{2}{x-1}}_{\text{remainder}}$

$\boxed{\text{OA: at } y=3x+1 \quad \text{VA: at } x=1}$

Oblique asymptote

3. For what value c is the following function continuous at $x = 9$?

$$f(x) = \begin{cases} \sqrt{x^2+19}-2 & \text{if } x < 9 \\ c & \text{if } x = 9 \\ x-1 & \text{if } x > 9 \end{cases}$$

We require

$$\lim_{x\to 9} f(x) = f(9).$$

Since $\lim_{x\to 9} f(x)$ EXISTS and EQUALS 9

we set $f(9) = \boxed{c=9}$