Math 1552 **Exam 1** Fall 2017

Instructor: Sal Barone

Name: KEY

GT username: ____________

1. No books or notes are allowed.
2. No electronic devices are allowed.
3. Show all work and fully justify your answer to receive full credit.
4. Please $\boxed{\text{BOX}}$ your answers.
5. The exam consists of 105 points but your score will be out of 100, there is a 5 pt. bonus question at the end.
6. Good luck!

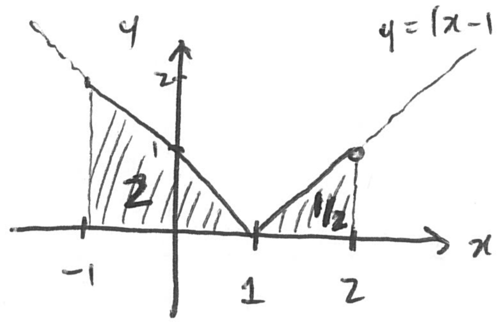
Page	Max. Possible	Points
1	38	
2	15	
3	26	
4	26	
Total	105	

1. **(a)** If F is an antiderivative of f, then: (5 pts.)

$$\int f(g(x))g'(x)\,dx = \boxed{F(g(x)) + C}$$

(b) Evaluate the integrals. (11 pts. each)

$$\int_{-1}^{2} |x-1|\,dx = \int_{-1}^{1} (1-x)\,dx + \int_{1}^{2} (x-1)\,dx = x - \tfrac{1}{2}x^2\Big|_{-1}^{1} + \tfrac{1}{2}x^2 - x\Big|_{1}^{2}$$



$$= \left[\left(1 - \tfrac{1}{2}(1)^2\right) - \left(-1 - \tfrac{1}{2}(-1)^2\right)\right]$$

$$+ \left[\left(\tfrac{1}{2}(2)^2 - 2\right) - \left(\tfrac{1}{2}(1)^2 - 1\right)\right]$$

$$= 2 + \tfrac{1}{2} = \boxed{2.5} \text{ ANS.}$$

$$\int \frac{1}{\sqrt{9-(1-x)^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1 - \frac{(1-x)^2}{9}}}\,dx$$

u-Sub Box

$$\boxed{\begin{aligned} u &= \frac{(1-x)}{3} \\ du &= -\frac{1}{3}\,dx \end{aligned}}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}} \cdot -3\,du$$

$$= -\sin^{-1}(u) + C = \boxed{-\sin^{-1}\left(\frac{1-x}{3}\right) + C} \text{ ANS.}$$

$$\frac{1}{2a}\int \frac{2a\,x}{(ax^2+b)^3}\,dx$$

u-Sub Box

$$\boxed{\begin{aligned} u &= ax^2 + b \\ du &= 2ax\,dx \end{aligned}}$$

$$= \frac{1}{2a}\int \frac{1}{u^3}\,du = \frac{1}{2a}\int u^{-3}\,du$$

$$= \frac{-1}{4a}u^{-2} + C = \frac{-1}{4a}(ax^2+b)^{-2} + C$$

$$= \boxed{\frac{-1}{4a(ax^2+b)^2} + C} \text{ ANS.}$$

2. The velocity of a particle is given by the formula $v(t) = 3t^2 + 6t - 2$, in meters per second. Evaluate the actual distance traveled between time $t = 1$ and $t = 2$ by taking a limit of Riemann sums using the general form of the definite integral. (15 pts.)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*).$$

ANY OTHER METHOD WILL NOT RECEIVE FULL CREDIT.

$$R_n = \sum_{k=1}^{n} \frac{2-1}{n}\left(3\left(1+\frac{k}{n}\right)^2 + 6\left(1+\frac{k}{n}\right) - 2\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n}\left(3\left(1+2k/n + \frac{k^2}{n^2}\right) + 6 + \frac{6k}{n} - 2\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n} 7 + \frac{12}{n^2}\sum_{k=1}^{n} k + \frac{3}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n}\cdot 7n + \frac{12}{n^2}\cdot\frac{n(n+1)}{2} + \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$R_n = 7 + 6\cdot\frac{n(n+1)}{n^2} + \frac{1}{2}\frac{n(2n+1)(n+1)}{n^3}$$

$$\lim_{n\to\infty} R_n = 7 + 6 + \frac{1}{2}(2)$$

ANS.

$$= 7 + 6 + 1 = \boxed{14}$$

Check.

$$\int_1^2 3t^2 + 6t - 2\,dt$$

$$= t^3 + 3t^2 - 2t \Big|_1^2$$

$$= (8 + 12 - 4) - (1 + 3 - 2)$$

$$= 16 - 2 = \boxed{14}$$

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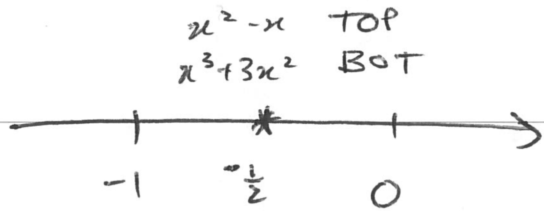
3. Find the area bounded by the curves $y = x^3 + 3x^2$ and $y = x^2 - x$. (12 pts.)

Set $y = y$



$y = f(x)$ $\left(-\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 = \frac{-1}{8} + \frac{3}{4}$

$\left(\frac{-1}{2}\right)^2 - \left(\frac{-1}{2}\right) = \frac{3}{4}$

$x^3 + 3x^2 = x^2 - x$

$x^3 + 2x^2 + x = 0$

$x(x^2 + 2x + 1) = 0$

$x(x+1)^2 = 0$

$x = 0, -1$

$$\int_{-1}^{0} \text{TOP} - \text{BOT}\; dx = \int_{-1}^{0} -x^3 - 2x^2 - x \; dx$$

$$= -\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 \Big|_{-1}^{0} = -\left(\frac{-1}{4} + \frac{2}{3} - \frac{1}{2}\right)$$

$$= \frac{3}{12} - \frac{8}{12} + \frac{6}{12} = \frac{9-8}{12} = \boxed{1/12} \text{ ANS.}$$

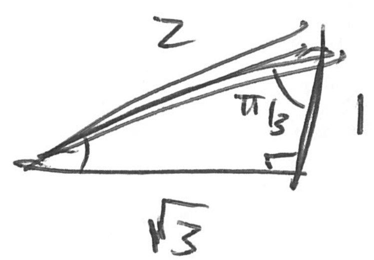
4. Solve the initial value problem: (14 pts.)

$$\frac{dy}{dx} = \frac{x + xy^2}{1 + x^2}, \quad y(0) = \sqrt{3}.$$

$$\int \frac{1}{1+y^2}\, dy = \frac{1}{2}\int \frac{2x}{1+x^2}\, dx$$

$$\tan^{-1}(y) = \frac{1}{2}\ln(1+x^2) + C$$

$$y = \tan\left(\frac{1}{2}\ln(1+x^2) + C\right)$$



$$\sqrt{3} = \tan\left(\frac{1}{2}\ln(1+0^2) + C\right) = \tan\left(\frac{1}{2}(0) + C\right) = \tan(C)$$

$C = \pi/3$

$$\boxed{y = \tan\left(\frac{1}{2}\ln(1+x^2) + \pi/3\right)} \text{ ANS.}$$

5. (a) Calculate $\frac{d}{dx}\left(\int_{2x}^{\ln e} e^{t^2}\,dt\right)$. (10 pts.)

$$= \frac{d}{dx}\left[-\int_{\ln e}^{2x} e^{t^2}\,dt\right]$$

$$= -e^{(2x)^2}\cdot 2 - 0$$

$$= \boxed{-2e^{4x^2}} \text{ ANS.}$$

(b) On a walk around the block, you walk with a velocity of $v(t) = \frac{e^{2t}}{5+e^{2t}}$ feet per second. What was your average velocity for the first $\ln 5$ seconds of your walk? (11 pts.)

$$\frac{1}{(\ln 5 - 0)}\int_0^{\ln 5}\frac{2e^{2t}}{5+e^{2t}}\,dt = \frac{1}{2\ln 5}\ln(5+e^{2t})\Big|_0^{\ln 5}$$

$$= \frac{1}{2\ln 5}\left(\ln(5+e^{2\cdot\ln 5}) - \ln(5+e^{2\cdot 0})\right)$$

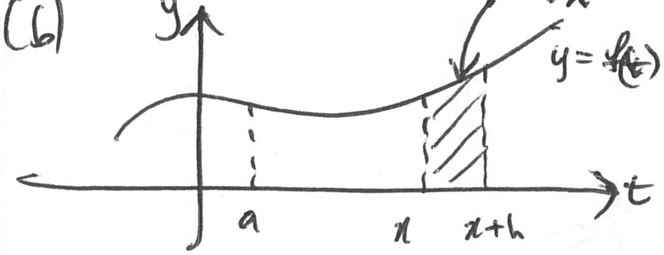
$$= \frac{1}{2\ln 5}\left(\ln(5+25)-\ln(6)\right) = \frac{1}{2}\cdot\frac{\ln(\cancel{30/6})}{\cancel{\ln(5)}} \; 1 = \boxed{\frac{1}{2}} \text{ ANS.}$$

Bonus: *(a)* State the Fundamental Theorem of Calculus (Part I). (2 pts.)

(b) Draw a picture which illustrates the relationship between $\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$ and $f(x)$, or explain in words or mathematically. (3 pts.)

(a) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

(b) $\int_x^{x+h} f(t)\,dt = F(x+h) - F(x)$



$$\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$= F'(x) = \boxed{f(x)}$$