Math 1552 **Exam 2** Summer-SOUP 2018

Instructor: Sal Barone

Name: Key

GT username: ____________________

1. No books or notes are allowed.
2. No electronic devices are allowed.
3. Show all work and fully justify your answer to receive full credit.
4. Please $\boxed{\text{BOX}}$ your answers.
5. The exam consists of 105 points but your score will be out of 100, there is a 5 pt. bonus question at the end.
6. Good luck!

Page	Max. Possible	Points
1	25	
2	39	
3	24	
4	17	
Total	105	

1. Solve the separable differential equation. (15 pts.)

$$\frac{dy}{dx} = xe^{x/3}(4y^2+1).$$

$$\int \frac{1}{4y^2+1}\,dy = \int xe^{x/3}\,dx = 3xe^{x/3} - \int 3e^{x/3}\,dx = 3xe^{x/3} - 9e^{x/3} + C$$

IBP

$u = x$ $dv = e^{x/3}\,dx$

$du = dx$ $v = 3e^{x/3}$

$y = \frac{1}{2}\tan\theta$

$dy = \frac{1}{2}\sec^2\theta\,d\theta$

$$\int \frac{1}{\tan^2\theta+1}\cdot\frac{1}{4}\sec^2\theta\,d\theta = \int \frac{1}{4}\,d\theta$$

$= \theta/4 + C$

$= \frac{1}{4}\tan^{-1}(2y)$

Solve for y

$$\frac{1}{4}\tan^{-1}(2y) = 3xe^{x/3} - 9e^{x/3} + C$$

$$\tan^{-1}(2y) = 12xe^{x/3} - 36e^{x/3} + c$$

$$2y = \tan(12xe^{x/3} - 36e^{x/3} + c)$$

$$\boxed{y = \frac{1}{2}\tan(12xe^{x/3} - 36e^{x/3} + c)}$$

2. **(a)** Find a closed formula for the n-th term of the sequence in terms of n. Then, **(b)** find the limit of the sequence using L'Hopital's rule. (10 pts.)

$$\frac{3}{5}, \frac{-6}{11}, \frac{12}{17}, \frac{-24}{23}, \frac{48}{29}, \ldots$$

$$\boxed{a_n = \frac{(-1)^n \cdot 3\cdot 2^n}{5+6n};\ n \geq 0}$$

3. Integrate. (13 pts. each)

(a) $\int 5\sin^5(x/3)\cos^2(x/3)\,dx$ $= \int 5\left(\sin^2(x/3)\right)^2 \cdot \cos^2(x/3)\cdot \sin(x/3)\,dx$

$$= 5\int \left(1-\cos^2(x/3)\right)^2 \cos^2(x/3)\cdot \sin(x/3)\,dx$$

$u = \cos(x/3)$

$du = -1/3 \sin(x/3)\,dx$

$$= -15\int (1-u^2)^2 u^2\,du$$

$$= -15\int (1-2u^2+u^4)u^2\,du$$

$$= -15\int (u^2-2u^4+u^6)\,du$$

$$= -15\cdot\left(\frac{u^3}{3} - \frac{2\cdot u^5}{5} + \frac{u^7}{7}\right)+C$$

$$= \boxed{-5\cos^3(x/3) + 6\cos^5(x/3) + \frac{15}{7}\cos^7(x/3) + C}$$

(b) $\int \frac{\sqrt{x^2-9}}{x}\,dx$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sqrt{9\sec^2\theta - 9}}{\cancel{3\sec\theta}}\cdot \cancel{3\sec\theta}\tan\theta\,d\theta$$

$x/3 = \sec\theta$

$$= \int 3\tan^2\theta\,d\theta$$

$$= 3\int(\sec^2\theta - 1)\,d\theta$$

$$= 3\tan\theta - 3\theta + C$$

$$= \cancel{3}\boxed{\frac{\sqrt{x^2-9}}{\cancel{3}} - 3\sec^{-1}(x/3) + C}$$

(c) $\int \frac{x+2}{x^3+x}\,dx$

$$\frac{x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$A(x^2+1) + (Bx+C)x = x+2$$

$$x^2(A+B) + Cx + A = x+2$$

$A+B=0$ $\quad C=1$ $\quad A=2$

$B=-2$

$$= \int\left(\frac{2}{x} + \frac{-2x+1}{x^2+1}\right)dx$$

$$= \int\frac{2}{x}\,dx - \int\frac{2x}{x^2+1}\,dx + \int\frac{1}{x^2+1}\,dx$$

$$= \boxed{2\ln x - \ln(x^2+1) + \tan^{-1}x + C}$$

4. Evaluate the limit using L'Hopital's rule. (12 pts.)

$$\lim_{x\to\infty}(x+e^x)^{1/x}$$

Set $y = (x+e^x)^{1/x}$ Then $\ln(y) = \frac{1}{x}\cdot \ln(x+e^x)$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(x+e^x)}{x} \overset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{\frac{1}{x-e^x} *(1+e^x)}{1}$$

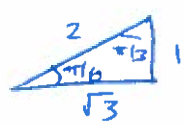
$$\overset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{+e^x}{1+e^x} \overset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{+e^x}{+e^x} = 1.$$

So $\ln(y) \to 1$.

Hence $y \to e^1 = \boxed{e}$

5. Find the approximate area bounded by the curve $y = \sin(x)$ over the interval $[0, \pi/2]$ using the trapezoid rule with $n = 3$ trapezoids. For up to 4 points, check your answer with FTC. (12 pts.)

$$\Delta x = \frac{b-a}{n} = \frac{\pi/2}{3} = \frac{\pi}{6}$$



$x_0 = 0$ $\sin(0) = 0$

$x_1 = \pi/6$ $\sin(\pi/6) = 1/2$

$x_2 = \pi/3$ $\sin(\pi/3) = \sqrt{3}/2$

$x_3 = \pi/2$ $\sin(\pi/2) = 1$

$$T_3 = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right)$$

$$= \frac{\pi}{12}\left(0 + 2\cdot\frac{1}{2} + 2\cdot\frac{\sqrt{3}}{2} + 1\right)$$

$$= \boxed{\frac{\pi}{12}\left(2+\sqrt{3}\right)}$$

6. Evaluate the improper integral. (12 pts.)

$$\int_1^\infty \frac{1}{x(2+\ln x)^2}\,dx$$

$u = 2 + \ln x$
$du = \frac{1}{x}\,dx$

when $x = 1$, $u = 2$

when $x = \infty$, $u = \infty$

$$= \int_2^\infty \frac{1}{u^2}\,du$$

$$= \left.\frac{-1}{u}\right|_2^\infty$$

$$= 0 - \frac{-1}{2}$$

$$= \boxed{1/2}$$

Bonus: (up to 5 pts) Suppose a population of bacterium, left unchecked, follows the rule that the current rate of growth of the population is proportional to the current population size. If the bacterium colony starts with 8 members, and after 2 days there are 400 bacterium, then what is the function which gives the number of bacterium in the colony after t days?

$$\frac{dP}{dt} = kP \implies P(t) = Ce^{kt}$$

$$P(0) = 8 \implies C = 0$$

$$P(2) = 400 \implies P(2) = 8 \cdot e^{2k} = 400$$

$$\implies e^{2k} = 50$$

$$\implies k = \ln(50)/2$$

ANS $\boxed{P(t) = 8 \cdot e^{\ln(50)/2 \cdot t}}$

or

$\boxed{P(t) = 8 \cdot (50)^{t/2}}$