KEY exam 3

1. Consider the sequence below.

$$a_n = \left(\frac{3n}{3n+1}\right)^{2n} \; ; \; n \geq 1.$$

(a) Find the limit of the sequence as n tends to infinity. (8 pts.)

$$\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} 2n \cdot \ln\left(\frac{3n}{3n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{3n}{3n+1}\right)}{1/2n}$$

$$\overset{\text{L'Hop}}{=} \lim_{n\to\infty} \frac{\frac{3n+1}{3n} * \frac{(3n+1)(3)-3n(3)}{(3n+1)^2}}{-1/2n^2} = \lim_{n\to\infty} \frac{-2n^2}{n(3n+1)} = -2/3$$

So $a_n \to \boxed{e^{-2/3}}$

(b) Does the series $\sum a_n$ converge or diverge? Give support for your answer. (6 pts.)

$\sum a_n$ diverges by the divergence test since $a_n \not\to 0$.

2. Find the value of each series, or show the series diverges. (12 pts. each)

(a) $\sum_{k=1}^{\infty} \frac{2^{k+1}5^k}{4^{2k+1}} = \sum_{k=1}^{\infty} \frac{2\cdot(2\cdot5)^k}{4\cdot(4^2)^k} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{10}{16}\right)^k$

$= \frac{1}{2}\cdot\frac{5}{8}\sum_{k=0}^{\infty}\left(\frac{5}{8}\right)^k = \frac{5}{16}\left(\frac{1}{1-5/8}\right)$

$= \frac{5}{16}\cdot\frac{1}{3/8} = \frac{5}{\cancel{16}_{2}}\cdot\frac{\cancel{8}}{3} = \boxed{5/6}$

(b) $\sum_{k=3}^{\infty} \frac{6}{k^2-k} = \sum_{k=3}^{\infty}\left(\frac{A}{k} + \frac{B}{k-1}\right)$

$A(k-1) + Bk = 6$

So $(A+B) = 0$

and $A = -6$

So $B = 6$

$= \sum_{k=3}^{\infty}\left(\frac{-6}{k} + \frac{6}{k-1}\right)$

$= \left(\cancel{\frac{-6}{3}} + \frac{6}{2}\right) + \left(\cancel{\frac{-6}{4}} + \cancel{\frac{6}{3}}\right) + \dots + \left(\frac{-6}{N} + \cancel{\frac{6}{N-1}}\right) + \dots$

tail goes to zero.

$= 6/2 = \boxed{3}$

b/c tail converges to zero.

3. Determine if each of the following series converge or diverge. Fully justify your answer with *(a)* complete work including a concluding statement which is a regular English statement which *(b)* specifies the test you used and *(c)* any required statements for the particular test you are using. (12 pts. each)

(a) $\sum_{n=1}^{\infty} \frac{n+\cos(\ln(n))}{n^2}$ $\geq \sum_{n=1}^{\infty} \frac{n-1}{n^2} \geq \sum_{n=1}^{\infty} \frac{(1/2)n}{n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$

diverges b/c harmonic.

By the direct comparison test

$\sum a_n \geq \sum b_n$ with $b_n = \frac{1}{2}\cdot\frac{1}{n}$,

the series $\boxed{\sum a_n \text{ diverges}}$ since $\sum b_n$ diverges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln(n))^{3/2}}$ Integral test

$$\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^{3/2}}\,du$$

converges by p-test w/ $p = 3/2 > 1$.

So by the integral test the series

$\boxed{\sum a_n \text{ converges}}$

(c) $\sum_{k=1}^{\infty} \frac{k^2 - 3k + 1}{(3k^2 - 1)(k^2 + k + 1)}$ limit comparison w/ $b_n = 1/k^2$

$\frac{a_n}{b_n} = \frac{k^2 - 3k + 1}{(3k^2-1)(k^2+k+1)} * k^2 \longrightarrow \frac{1}{3} = L$ is between $0 < L < \infty$

So $\sum a_n$ & $\sum b_n$ behave the same.

Since $\sum b_n$ converges by p-test w/ $p = 2 > 1$,

the series $\boxed{\sum a_n \text{ also converges}}$

4. Determine if the given alternating series converges absolutely, converges conditionally, or diverges. (12 pts.)

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n^2 - 1}}$$

$\boxed{\text{Converges conditionally}}$

since $\sum \frac{1}{\sqrt{n^2-1}}$ diverges by direct comparison

w/ $b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n} < \frac{1}{\sqrt{n^2-1}}$

and $\sum b_n$ diverges b/c harmonic

AND

$a_n \to 0$, pos, decreasing so by alt. series test $\sum (-1)^n a_n$ converges.

5. Find the radius and interval of convergence of the power series. (14 pts.)

$$\sum_{n=1}^{\infty} \frac{(3x+1)^n}{\sqrt{n}}$$

ratio ~~test~~

$$\frac{a_{n+1}}{a_n} = \frac{(3x+1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(3x+1)^n} = (3x+1)\cdot\sqrt{\frac{n}{n+1}}$$

$$\longrightarrow 3x+1 = L$$

need $|L| < 1$ so $|3x+1| < 1$

so $-1 < 3x+1 < 1$

so $-2 < 3x < 0$

$x \in (-2/3, 0)$

$$\boxed{\left[\frac{-2}{3}, 0\right) \quad R = 1/3}$$

check $x = -2/3$ $\sum \frac{(3\cdot -2/3+1)^n}{\sqrt{n}} = \sum (-1)^n \cdot \frac{1}{\sqrt{n}}$ conv. by alt. series so

$x = 0$ $\sum \frac{1}{\sqrt{n}}$ diverges by p-test w/ $p = 1/2 < 1$.

Bonus : (5 pts.) For each of the following, give an example of a sequence which satisfies the condition or say "impossible". No work is required.

(a) a_n which converges to e^2. $a_n = e^2 - \frac{1}{n}$

(b) b_n which converges to zero and $\sum b_n$ converges. $b_n = 1/n^2$

(c) c_n which converges to zero but $\sum c_n$ diverges. $c_n = 1/n$

(d) d_n a bounded sequence which diverges. $d_n = (-1)^n$

(e) e_n a sequence which is not monotone but converges. $e_n = \frac{(-1)^n}{n}$