Math 1552, Integral Calculus

Review for Test 2

Sections 8.4-8.5, 4.5, 8.8, 10.1-10.2

NOTE FROM THE INSTRUCTORS While the list of sections above does not include every integration technique, please note that students are expected to also understand how to integrate with u-substitutions, by parts, and using trig identities, as these techniques may be needed in order to evaluate integrals from the above listed sections. You will find some review problems from previous sections incorporated in the problems below.

1. Content Recap

(a) To apply L'Hopital's rule, the limit must have the indeterminate form ____________ or ____________.

(b) An integral $\int_a^b f(x)dx$ is *improper* if at least one of the limits of integration is ____________, or if there is a ____________ ____________ on the interval $[a, b]$.

(c) If we would evaluate an integral using *trig substitution*, the integral should contain an expression of one of these forms: ____________, ____________, or ____________.

Write out the trig substitution you would use for each form listed above.

(d) To use the method of *partial fractions*, we must first factor the denominator completely into ____________ or ____________ ____________ terms.

In the partial fraction decomposition, if the term in the denominator is raised to the kth power, then we have ____________ partial fractions.

For each linear term, the numerator of the partial fraction will be ____________.

For each irreducible quadratic term, the numerator will be ____________.

(e) Define the least upper bound and greatest lower bound of a sequence.

(f) What does it mean for a sequence to be monotonic?

(g) A sequence **converges** if ____________ and **diverges** if ____________.

(h) If a sequence is ____________ and ____________, then it converges.

(i) A geometric series has the general form ____________.
The series converges when ____________ and diverges when ____________.

(j) The harmonic series has the general form ____________, and it always ____________!

(k) To find the sum of a telescoping series, we should first break it into ____________ ____________.

(l) The series $\sum a_n$ diverges if the limit is NOT equal to ____________.

Problems from Recitation Worksheets

2. Determine if the following statements below are always true or sometimes false.

(a) If an integral contains the term $a^2 + x^2$, we should use the substitution $x = a\sec\theta$.

(b) The expression $\tan\left(\sin^{-1}(x)\right)$ cannot be simplified.

(c) When using a trig substitution with a term of the form $a^2 - x^2$, we could use either $x = a\sin\theta$ or $x = a\cos\theta$ and obtain equivalent answers (that may differ only by a constant).

(d) If we use the trig substitution $x = \sin\theta$, then it is possible that $\sqrt{1-x^2} = -\cos\theta$.

(e) The partial fraction decomposition of $\frac{x}{(x+3)^2}$ is $\frac{A}{x+3} + \frac{B}{(x+3)^2}$.

(f) $\int \frac{dx}{(x+3)^2} = \ln(x+3)^2 + C$.

(g) The integral $\int \frac{x}{x^2-9}dx$ could be best evaluated using the method of partial fractions.

(h) The integral $\int \frac{dx}{x(x^4+1)}$ cannot be evaluated using the method of partial fractions.

(i) $\lim_{x\to\infty} xe^x$ has the indeterminate form ∞^∞.

(j) $\lim_{x\to 0^+}(\cos x)^{\frac{1}{x}}$ has the indeterminate form 1^∞.

(k) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = 2e$.

(l) When evaluating a limit using L'Hopital's rule, we first need to find $\left(\frac{f}{g}\right)'$.

(m) If f has a vertical asymptote at $x = a$, then $\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$.

(n) $\int_{-1}^{1} \frac{1}{x}dx = 0$.

(o) Saying that an improper integral converges means that the integral must evaluate to a finite number.

(p) Indefinite integrals can be improper.

(q) If $\{a_n\}$ is bounded, then it converges.

(r) If $\{a_n\}$ converges, then it is monotonic.

(s) An unbounded sequence diverges.

(t) If $\{a_n\}$ diverges, then $\lim_{n\to\infty} a_n = \infty$.

(u) The sequence $\{\frac{1}{n}\}$ converges to 0.

(v) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum a_n$ converges.

(w) If $\lim_{n\to\infty} a_n = 0$, then the sequence $\{a_n\}$ converges.

(x) The series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

(y) The sum of two divergent series also diverges.

(z) The series $\sum_{n=1}^{\infty} r^n$ converges to $\frac{1}{1-r}$ if $|r| < 1$.

3. Evaluate the following integrals using any method we have learned so far: u-substitutions, integration by parts, integrating trig functions, trigonometric substitutions, or partial fractions.

(a) $\int \frac{x^2}{(x^2+4)^{3/2}} dx$

(b) $\int (x^2+1)e^{2x} dx$

(c) $\int \frac{\sqrt{1-x^2}}{x^4} dx$

(d) $\int \frac{dx}{e^x\sqrt{e^{2x}-9}}$

(e) $\int \sin^2(x)\cos^2(x) dx$

(f) $\int \frac{x+3}{(x-1)(x^2-4x+4)} dx$

(g) $\int x^5 \ln(x) dx$

(h) $\int \frac{x+4}{x^3+x} dx$

(i) $\int \sqrt{25-x^2} dx$

(j) $\int \frac{x-1}{(x+1)^3} dx$

(k) $\int \frac{x+2}{x+1} dx$

(l) $\int \frac{x+1}{x^2(x-1)} dx$

4. Evaluate the following limits using L'Hopital's Rule.

(a) $\lim_{x\to 0^+} [x(\ln(x))^2]$

(b) $\lim_{x\to\infty} (x+e^x)^{2/x}$

(c) $\lim_{x\to\frac{\pi}{2}} \left[\frac{\ln(\sin x)}{(\pi-2x)^2}\right]$

5. Evaluate the improper integrals if they converge, or show that the integral diverges.

(a) $\int_0^3 \frac{x}{(x^2-1)^{2/3}} dx$

(b) $\int_0^\infty x^2 e^{-2x} dx$

(c) $\int_1^4 \frac{dx}{x^2-5x+6}$

6. For each sequence below, find the l.u.b. and g.l.b., and determine if the sequence is monotonic.

(a) $\{\sin(n\pi)\}$

(b) $\left\{(-1)^{n+1}\frac{1}{5^n}\right\}$

(c) $\left\{\frac{n+1}{n}\right\}$

7. Determine whether or not each sequence converges. If so, find the limit.

(a) $\left\{\frac{2n^2}{\sqrt{9n^4+1}}\right\}$

(b) $\left\{\left(1-\frac{1}{8n}\right)^n\right\}$

(c) $\left\{\frac{n!}{e^n}\right\}$

(d) $\left\{\left(\frac{n}{n+5}\right)^n\right\}$

8. Use series to write the repeating decimal 0.31313131... as a rational number.

9. Find the sum of each convergent series below, or explain why the series diverges.

(a) $\sum_{k=7}^{\infty} \frac{1}{(k-3)(k+1)}$

(b) $\sum_{k=0}^{\infty} (-1)^k$

(c) $\sum_{k=2}^{\infty} \frac{2^k+1}{3^{k+1}}$

(d) $\sum_{k=1}^{\infty} \frac{5k^2+8}{7k^2+6k+1}$

(e) $\sum_{n=0}^{\infty} \left(\frac{2}{3^n} + \frac{(-1)^n}{6^n}\right)$

Additional Test Review Problems

10. Determine if each statement below is always true or sometimes false.

(a) $\lim_{x\to 2} \frac{x-2}{x^2+x-6}$ is of an indeterminate form.

(b) $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = e$

(c) The integral $\int x^3\sqrt{1-x^2}\,dx$ can be evaluated by trigonometric substitution by setting $x = \sin x$.

(d) $\sin(\cos^{-1}(x)) = \tan(x)$.

(e) For the rational expression $\frac{x}{(x+10)(x-10)^2}$, the partial fraction decomposition is of the form $\frac{A}{x+10} + \frac{B}{(x-10)^2}$.

(f) For the rational expression $\frac{2x+3}{x^2(x+2)^2}$, the partial fraction decomposition is of the form $\frac{A}{x^2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$.

(g) If a sequence $\{a_n\}$ converges to a finite number L, then either the least upper bound or the greatest lower bound for $\{a_n\}$ is equal to L.

(h) If a sequence $\{a_n\}$ has both an upper bound and a lower bound, then $\{a_n\}$ converges.

(i) If a sequence $\{a_n\}$ does not have an upper bound, then $\{a_n\}$ diverges.

(j) The sum of two convergent geometric series is also convergent.

(k) The difference of two divergent series is also divergent.

(l) If $\sum a_n$ converges, then $\lim_{n\to 0} a_n = 0$.

(m) The integral $\int_{-1}^{1} \frac{1}{x^2}\,dx$ can be evaluated using the Fundamental Theorem of Calculus.

(n) The integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p \geq 1$.

11. Evaluate each integral using any method we have learned.

(a) $\int \frac{2x+1}{x^2-7x+12}\,dx$

(b) $\int \frac{8\,dx}{x^2\sqrt{4-x^2}}$

(c) $\int \frac{8\,dx}{(4x^2+1)^2}$

(d) $\int \frac{1}{(x+1)(x^2+1)}\,dx$

12. Use L'Hopital's rule to evaluate the following limits.

(a) $\lim_{x\to\infty} (\ln x)^{\frac{1}{x^2+1}}$

(b) $\lim_{x\to 0^+} (\ln x)^x$

(c) $\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]$
(d) $\lim_{x\to\infty}\left[\cos\left(\frac{1}{x}\right)\right]^x$

13. Find values of a and b so that

$$\lim_{x\to 0}\frac{\cos(ax)-b}{2x^2}=-4.$$

14. Determine whether the sequences converge or diverge. Find the limit of each convergent sequence.
(a) $\left\{\left(1-\frac{1}{n^2}\right)^n\right\}$
(b) $\left\{(10n)^{\frac{1}{n}}\right\}$
(c) $\left\{\frac{2n+(-1)^n}{4+3n}\right\}$

15. Find a formula for the nth term of the sequence. Then, determine whether the sequences converge or diverge. Find the limit of each convergent sequence.
(a) $\{1,-1,1,-1,1,-1,\ldots\}$
(b) $\{\sqrt{5}-\sqrt{4},\sqrt{6}-\sqrt{5},\sqrt{7}-\sqrt{6},\sqrt{8}-\sqrt{7},\sqrt{9}-\sqrt{8},\sqrt{10}-\sqrt{9},\ldots\}$
(c) $\left\{\sin\left(\frac{\sqrt{2}}{5}\right),\sin\left(\frac{\sqrt{3}}{10}\right),\sin\left(\frac{\sqrt{4}}{17}\right),\sin\left(\frac{\sqrt{5}}{26}\right),\sin\left(\frac{\sqrt{6}}{37}\right),\sin\left(\frac{\sqrt{7}}{50}\right),\ldots\right\}$

16. Determine if each integral below converges or diverges, and evaluate the convergent integrals.
(a) $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$
(b) $\int_7^\infty \frac{dx}{x^2-x}$
(c) $\int_6^\infty \frac{3t^2}{\sqrt{t^3-8}}\,dt$
(d) $\int_2^6 \frac{3t^2}{\sqrt{t^3-8}}\,dt$
(e) $\int_2^\infty \frac{3t^2}{\sqrt{t^3-8}}\,dt$

17. Determine if each infinite series converges or diverges. If it converges, find the sum.
(a) $\sum_{n=0}^\infty e^{-3n}$
(b) $\sum_{n=1}^\infty \frac{4^n+5^n}{9^{n-1}}$
(c) $\sum_{n=0}^\infty \ln\left(\frac{n+5}{n+6}\right)$
(d) $\sum_{n=0}^\infty \cos(5\pi n)$
(e) $\sum_{n=1}^\infty \left(\frac{1}{\sqrt{n+3}}-\frac{1}{\sqrt{n+5}}\right)$

ANSWERS

2. (c), (e), (h), (j), (m), (o), (s), (u), (w), (x), (y) are true

3. (a) $\ln|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}|-\frac{x}{\sqrt{x^2+4}}+C$ (b) $\frac{1}{2}(x^2+1)e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}+C$

(c) $-\frac{1}{3}\cdot\frac{(1-x^2)^{3/2}}{x^3}+C$ (d) $\frac{\sqrt{e^{2x}-9}}{9e^x}+C$ (e) $\frac{x}{8}-\frac{1}{32}\sin(4x)+C$

(f) $4\ln\left|\frac{x-1}{x-2}\right|-\frac{5}{x-2}+C$ (partial fractions)

(g) $\frac{x^6\ln x}{6}-\frac{x^6}{36}+C$ (by parts)

(h) $4\ln|x|-2\ln(x^2+1)+\tan^{-1}(x)+C$ (partial fractions)

(i) $\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)+\frac{x\sqrt{25-x^2}}{2}+C$ (trig sub)

(j) $-\frac{1}{x+1}+\frac{1}{(x+1)^2}+C$ (k) $x+\ln|x+1|+C$

(l) $-2\ln|x|+\frac{1}{x}+2\ln|x-1|+C$

4. (a) 0, (b) e^2, (c) $-\frac{1}{8}$

5. (a) $\frac{9}{2}$, (b) $\frac{1}{4}$, (c) diverges

6. (a) l.u.b.=g.l.b.=0 (b) l.u.b.=$\frac{1}{5}$ and g.l.b.=$-\frac{1}{25}$

(c) l.u.b.=2 and g.l.b.=1

7. (a) $\frac{2}{3}$ (b) $e^{-1/8}$ (c) diverges (d) $\frac{1}{e^5}$

8. $\frac{31}{99}$

9. (a) ≈ 0.1899 (b) diverges (c) $\frac{1}{2}$ (d) diverges (e) $3\frac{6}{7}$

10. (c), (i), (j), (l) are true

11. (a) $-7\ln|x-3|+9\ln|x-4|+C$, (b) $\frac{-2\sqrt{4-x^2}}{x}+C$

(c) $2\tan^{-1}(2x)+\frac{4x}{4x^2+1}+C$, (d) $\frac{1}{2}\ln|x+1|+\frac{1}{2}\arctan x-\frac{1}{4}\ln|x^2+1|+C$

12. (a) 1, (b) 1, (c) 0, (d) 1

13. $a=\pm 4$, $b=1$

14. (a) 1, (b) 1, (c) $\frac{2}{3}$

15. (a) $\left\{(-1)^{n+1}\right\}$ and diverges, (b) $\left\{\sqrt{n+4}-\sqrt{n+3}\right\}$ and converges to 0

(c) $\left\{\sin\left(\frac{\sqrt{n+1}}{1+(n+1)^2}\right)\right\}$ and converges to 0.

16. (a) 1; (b) $\ln\left(\frac{6}{7}\right)$; (c) diverges; (d) $8\sqrt{13}$; (e) diverges

17. (a) $\frac{e^3}{e^3-1}$, (b) $\frac{369}{20}$, (c) diverges, (d) diverges, (e) $\frac{1}{2}+\frac{1}{\sqrt{5}}$