1. Evaluate the integrals. (10 pts. each)

(a) $\int 8x\sec^2(4x)\,dx = 8x\cdot\frac{1}{4}\tan(4x) - \int \frac{1}{4}\tan(4x)\cdot 8\,dx$

IBP

$\boxed{\begin{array}{ll} u = 8x & dv = \sec^2(4x)\,dx \\ du = 8\,dx & v = \frac{1}{4}\tan(4x)\end{array}}$

$= 2x\tan(4x) - 2\int \tan(4x)\,dx$

$= 2x\tan(4x) - 2\int \frac{\sin(4x)}{\cos(4x)}\,dx$

u-sub

$\boxed{\begin{array}{l} u = \cos(4x) \\ du = -4\sin(4x)\,dx\end{array}}$

$= 2x\tan(4x) - 2\ln(\cos(4x))\cdot\frac{-1}{4} + C$

ANS.

$= \boxed{2x\tan(4x) + \frac{1}{2}\ln(\cos 4x) + C}$

(b) $\int \ln(3x)\,dx$

IBP

$\boxed{\begin{array}{ll} u = \ln(3x) & dv = dx \\ du = \frac{1}{x}\,dx & v = x\end{array}}$

$= x\ln(3x) - \int x\cdot\frac{1}{x}\,dx$

$= x\ln(3x) - \int 1\,dx$

ANS.

$= \boxed{x\ln 3x - x + C}$

2. Evaluate the improper integral. (10 pts.)

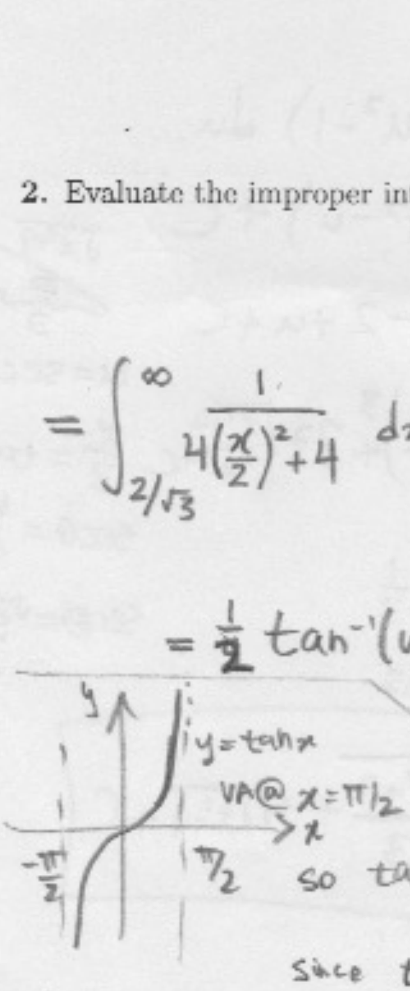
$$\int_{2/\sqrt{3}}^{\infty} \frac{1}{x^2+4}\,dx$$

u-sub

$\boxed{u = \frac{x}{2} \quad du = \frac{1}{2}\,dx}$

$$= \int_{2/\sqrt{3}}^{\infty} \frac{1}{4\left(\frac{x}{2}\right)^2+4}\,dx = \frac{1}{4}\int_{2/\sqrt{3}}^{\infty} \frac{1}{\left(\frac{x}{2}\right)^2+4}\,dx = \frac{1}{4}\int_{*}^{*} \frac{1}{u^2+4}\cdot 2\,du$$

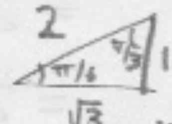
$$= \frac{1}{2}\tan^{-1}(u)\Big|_{*}^{*} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_{2/\sqrt{3}}^{\infty} = \lim_{N\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{N}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{2/\sqrt{3}}{2}\right)$$



so $\tan^{-1}(N) \to \pi/2$ as $N\to\infty$

since $\tan(x) \to \infty$ as $x \to \pi/2$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\tan^{-1}(1/\sqrt{3})$$



ANS.

$$= \pi/4 - \frac{1}{2}\cdot\pi/6 = \frac{\pi}{4} - \frac{\pi}{12} = \boxed{\pi/6}$$

3. Evaluate the integrals. (10 pts. each)

(a) $\int_0^{\pi/18} \sin^5(9x)\,dx = \int_0^{\pi/18} \left(\sin^2(9x)\right)^2 \cdot \sin(9x)\,dx$

$= \int_0^{\pi/18} \left(1-\cos^2(9x)\right)^2 \sin(9x)\,dx$

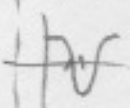
$= \int_0^{\pi/18} \left(1-2\cos^2(9x)+\cos^4(9x)\right)\sin(9x)\,dx$

u-sub

$u=\cos(9x)$

$du=-9\sin(9x)\,dx$

when $x=\pi/18$



$u=\cos(9\cdot\pi/18)$

$u=\cos(\pi/2)=0$

when $x=0$

$u=\cos(0)=1$

to get new limits

$= \frac{-1}{9}\int_1^0 1-2u^2+u^4\,du$

$= \frac{-1}{9}\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)\Big|_1^0$

ANS

$= \frac{-1}{9}(0) - \frac{-1}{9}\left(1-\frac{2}{3}+\frac{1}{5}\right) = 0+\frac{1}{9}\left(\frac{15-10+3}{15}\right) = \frac{1}{9}\left(\frac{8}{15}\right) = \boxed{8/135}$

(b) $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

$\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

trig sub

$x=3\tan\theta$

$x^2+9=9\tan^2\theta+9=9\sec^2\theta$

$dx=3\sec^2\theta\,d\theta$

$= \int \frac{(3\tan\theta)^3}{\sqrt{9\sec^2\theta}}\,3\sec^2\theta\,d\theta$

$= \int \frac{27\tan^3\theta}{\cancel{3}\sec\theta}\,\cancel{3}\sec^2\theta\,d\theta$

$= 27\int \tan^3\theta\sec\theta\,d\theta$

$= 27\int \tan^2\theta\,\sec\theta\tan\theta\,d\theta$

u-sub

$u=\sec\theta$

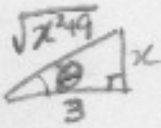
$du=\sec\theta\tan\theta\,d\theta$

$= 27\int (\sec^2\theta-1)\cdot\sec\theta\tan\theta\,d\theta$

$= 27\int (u^2-1)\,du$

$= 27\left(\frac{1}{3}u^3-u\right)+C$

$= 9u^3-27u+C$



$u=\sec\theta$

$= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 27\cdot\frac{\sqrt{x^2+9}}{3}+C$

$\frac{x}{3}=\tan\theta$

$\sec\theta=\frac{hyp}{adj}$

$\sec\theta=\frac{\sqrt{x^2+9}}{3}$

$\frac{9}{3^3} = \frac{3^2}{3^3} = \frac{1}{3}$

ANS

$\boxed{\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 9\sqrt{x^2+9}+C}$

4. Evaluate the integral using a partial fractions decomposition. (10 pts.)

$$\int \frac{x+4}{x^2+6x-16}\,dx$$

$$\frac{x+4}{(x+8)(x-2)} = \frac{A}{x+8} + \frac{B}{x-2}$$

$A(x-2)+B(x+8) = x+4$

$(A+B)x + (-2A+8B) = x+4$

$A+B = 1$ (not 0!!!)

$-2A+8B = 4$

$-2A+8(1-A) = 4$

$-10A+8=4$ $A = 4/10 = 2/5$ $B = 3/5$

$$= \int \left(\frac{2/5}{x+8} + \frac{3/5}{x-2}\right) dx$$

Ans.

$$= \boxed{\frac{2}{5}\ln|x+8| + \frac{3}{5}\ln|x-2| + C}$$

5. Determine whether or not the improper interval converges or diverges. You do not need to compute the value of the integral, but ou must clearly and fully justify your answer to receive full credit on this problem. (8 pts.)

$$\int_1^\infty \frac{x-1}{x^2+1}\,dx$$

Guess: diverge b/c compare to $\frac{1}{x} \approx \frac{x-1}{x^2+1}$ when $x \gg 0$.

and $\int_1^\infty \frac{1}{x}\,dx = \lim_{N\to\infty} \ln(N) - \ln(1) = +\infty$ DNE diverges.

$$(*)\ \int_1^\infty \frac{x-1}{x^2+1}\,dx = \int_1^\infty \left(\frac{x}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

Now,

$\int_1^\infty \frac{x}{x^2+1}\,dx$ diverges b/c $\frac{x}{x^2+1} > \frac{x}{2x^2} = \frac{1}{2x}$ and $\int_1^\infty \frac{1}{2x}\,dx$ diverges.

So $(*)$ diverges too. (Note. $\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{N\to\infty} \tan^{-1}(N) - \tan^{-1}(1) = \pi/2 - \pi/4 = \pi/4$ converges)

6. Solve the initial value problem. (10 pts.)

$$\frac{dy}{dx} + 5xy = 4x,\ y(0) = -3$$

$v(x) = e^{\int P(x)\,dx}$

$P(x) = 5x$

$Q(x) = 4x$

$v(x) = e^{\int 5x\,dx}$

$v(x) = e^{5x^2/2}$

$y = \frac{1}{v(x)} \int v(x)\,Q(x)\,dx$

$y = e^{-5x^2} \int e^{5x^2} \cdot 4x\,dx$

u-sub

$\boxed{u = 5x^2/2 \quad du = 5x\,dx}$

$y = e^{-\frac{5x^2}{2}} \cdot \frac{4}{5} \int e^u\,du$

$y = \frac{4}{5} e^{-\frac{5x^2}{2}} \left(e^{\frac{5x^2}{2}} + C\right)$ & when $x = 0$, $y = -3$ so

$-3 = \frac{4}{5} e^0 (e^0 + C)$ $\quad -3 = \frac{2}{5} \cdot C \quad C = \frac{-15}{2}$

Ans.

$y = \frac{4}{5} e^{-\frac{5x^2}{2}} \left(e^{\frac{5x^2}{2}} - \frac{15}{2}\right)$ or $\boxed{y = \frac{4}{5} - \frac{15}{2} e^{-5x^2/2}}$

7. Find the limit. (8 pts.)

$$\lim_{x \to 0} \frac{12x^2}{\cos x - 1}$$

$\lim_{x \to 0} \frac{12x^2}{\cos x - 1} \overset{\text{L'Hop } 0/0}{=} \lim_{x \to 0} \frac{24x}{-\sin x} \overset{\text{L'Hop } 0/0}{=} \lim_{x \to 0} \frac{24}{-\cos x} = \boxed{-24}$

8. Find a closed formula for the given sequence. Be sure to include the range of n-values in your answer. (3 pts. each)

(a) 0, 1, 0 , 1, 0, ...

$$\boxed{a_n = \frac{(-1)^n+1}{2} \quad n \geq 1}$$

(b) 3, 8, 15, 24, 35, 48, ...

$$\boxed{a_n = n^2 - 1;\ n \geq 2} \quad \text{or} \quad \boxed{a_n = n^2 + 2n;\ n \geq 1}$$

9. Find the limit of the sequences. (4 pts. each)

(a) $a_n = \left(\frac{n+2}{6n}\right)\left(2 - \frac{4}{n}\right)$.

$$\lim_{n\to\infty} \left(\frac{n+2}{6n}\right)\left(2-\frac{4}{n}\right) = \lim_{n\to\infty} \left(\frac{n+2}{6n}\right)\left(\frac{2n-4}{n}\right)$$

$$= \lim_{n\to\infty} \frac{2n^2 \cancel{-4n} \cancel{+4n} - 8}{6n^2} = 2/6 = \boxed{1/3}$$

(b) $a_n = \left(1 - \frac{4}{n}\right)^n$. *Hint: use properties of* $\ln(x)$.

$$y_n = \ln(a_n) = n \ln\left(1 - \frac{4}{n}\right)$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} n \cdot \ln\left(1-\frac{4}{n}\right) = \lim_{n\to\infty} \frac{\ln(1-4/n)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1-4/n} \cdot 4/n^2}{-1/n^2}$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{-4}{1-4/n} = -4$$

Since $\lim_{n\to\infty} \ln(a_n) = -4$ $\quad a_n \to e^{-4}$ as $n\to\infty$

(idea: if $\ln(X) = -4$ then $X = e^{-4}$)