1. Express the repeating decimal as the ratio of two integers. Simplify if necessary. (10 pts.)

$$4.2\overline{35} = 4.2353535\ldots$$

$$4.2 + .0353535 = 4.2 + \frac{35}{10^3} + \frac{35}{10^5} + \frac{35}{10^7} + \cdots$$

$$= 4.2 + 35 \sum_{n=0}^{\infty} \frac{1}{10^{2n+3}}$$

$$= 4.2 + 35 \cdot \frac{1}{10^3} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n$$

$$= 4.2 + \frac{35}{1000}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{42}{10} + \frac{35}{1000}\left(\frac{1}{99/100}\right)$$

$$= \frac{42}{10} + \frac{35}{1000}\left(\frac{100}{99}\right)$$

$$42 * 99 = 42 \cdot 100 - 42$$
$$= 4200 - 42$$
$$= 4158$$

$$= \frac{1}{10}\left(\frac{42 \cdot 99 + 35}{99}\right) = \frac{1}{10}\left(\frac{4158 + 35}{99}\right)$$

$$= \boxed{4193/990}$$

2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)}$. (10 pts.)

$$\frac{9}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2}$$

$$(3n-1)B + (3n+2)A = 9$$

$$\begin{cases} 3B + 3A = 0 \\ 2A - B = 9 \end{cases}$$

$$\begin{cases} A = -B \\ 2A + A = 9 \end{cases}$$

$$\begin{cases} A = 3 \\ B = -3 \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)} = \sum_{n=1}^{\infty}\left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N}\left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$$

$$= \lim_{N\to\infty}\left(\frac{3}{2} - \cancel{\frac{3}{5}} + \cancel{\frac{3}{5}} - \cancel{\frac{3}{8}} + \cancel{\frac{3}{8}} - \cancel{\frac{3}{11}} + \cdots + \cancel{\frac{3}{3N-1}} - \frac{3}{3N+2}\right)$$

$$= \lim_{N\to\infty} \frac{3}{2} - \frac{3}{3N+2}$$

$$= 3/2$$

3. Find the sum of the series $\sum_{n=0}^{\infty} e^{\frac{-3n}{2}}$ or state that the series diverges and justify your answer.

(10 pts.)

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r|<1$$

$$\sum_{n=0}^{\infty} (e^{-3/2})^n = \boxed{\frac{1}{1-e^{-3/2}}} \quad \text{b/c} \quad |e^{-3/2}| = \frac{1}{e^{3/2}} < 1$$

4. Determine whether the given series converges or diverges. Justify your answer. (10 pts.)

$$\sum_{n=1}^{\infty} \frac{4}{2\sqrt{n} + \sqrt[3]{n}}$$

Several answers possible.

Soln using limit comparison w/ $b_n = 1/\sqrt{n}$ $\qquad \frac{1}{3} - \frac{1}{2} = -1/6$

$$\frac{a_n}{b_n} = \frac{4/2\sqrt{n}+\sqrt[3]{n}}{1/\sqrt{n}} = \frac{4\sqrt{n}}{2\sqrt{n}+\sqrt[3]{n}} = \frac{4\cdot\sqrt{n}}{2\sqrt{n}+\sqrt[3]{n}} \cdot \frac{1/\sqrt{n}}{1/\sqrt{n}} = \frac{4}{2+n^{-1/6}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4}{2+n^{-1/6}} = \frac{4}{2} = 2$$

So $\sum a_n$ diverges $\Leftrightarrow$ $\sum b_n$ diverges.

And $\sum \frac{1}{\sqrt{n}}$ diverges by 2 p-test w/ $p = 1/2$.

So the series DIVERGES

5. Determine whether the given series converges or diverges. Justify your answer. (10 pts.)

$$\sum_{n=1}^{\infty} \frac{3n}{4n-7}$$

Diverges by n-th term test since

$$\lim_{n\to\infty} \frac{3n}{4n-7} = 3/4 \neq 0.$$

$$\frac{3(n+1)}{3n} \quad \frac{4(n)-7}{4(n+1)-7}$$

$$\frac{3(1+\frac{1}{n})}{3} \quad \frac{4-\frac{7}{n}}{4(1+\frac{1}{n})-\frac{7}{n}} = 1$$

6. Determine whether the given series converges or diverges. Justify your answer. (10 pts.)

$$\sum_{n=1}^{\infty} \frac{3\sqrt{n}+1}{4n+n^2}$$

~~Converg ... ~~

Soln using direct comparison test. (guess $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ converges)

$$\frac{(3\sqrt{n}+1)\,n^2}{4n+n^2} = \frac{3\sqrt{n}+1}{4\frac{1}{n}+1} \to \infty$$

$$\frac{3\sqrt{n}+1}{4n+n^2} \leq \frac{3\sqrt{n}}{\cancel{4n}+n^2} = \frac{3\sqrt{n}}{5n^2} = \frac{3}{5n^{3/2}}$$

and $\sum \frac{3}{n^{3/2}}$ converges by p-test w/ $p = 3/2$.

So $\sum a_n \leq \sum b_n \leftarrow$ converges

So $\sum a_n$ converges

root:

$$\frac{\sqrt[n]{3\sqrt{n}+1}}{\sqrt[n]{n}\,\sqrt[n]{4+n}} \to \frac{1}{1}$$

ratio:

$$\frac{3\sqrt{n+1}+1}{3\sqrt{n}+1} \cdot \frac{4n+n^2}{4(n+1)+(n+1)^2} \to 1 \cdot 1 = 1$$

7. Find a Taylor series, at $x = 0$, representation of the function $f(x) = \frac{3x}{1+2x}$. (10 pts.)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-2x)} = \sum_{n=0}^{\infty} (-2x)^n$$

$$3x \cdot \frac{1}{1-(-2x)} = 3x \sum_{n=0}^{\infty} (-2x)^n$$

$$\frac{3x}{1+2x} = \sum_{n=0}^{\infty} 3 \cdot (-1)^n \cdot 2^n \cdot x^n$$

8. Determine the interval and radius of convergence of the power series. (10 pts.)

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt{n} 4^n}$$

$$\left| \frac{3^{n+1} x^{n+1} / \sqrt{n+1}\, 4^{n+1}}{3^n x^n / \sqrt{n}\, 4^n} \right| = \frac{3}{4} \frac{\sqrt{n}}{\sqrt{n+1}} x \longrightarrow \left|\frac{3}{4} x\right|$$

$$\left|\frac{3}{4} x\right| < 1 \iff x \in \left(-\frac{4}{3}, \frac{4}{3}\right)$$

If $x = 4/3$

$$\sum_{n=1}^{\infty} \frac{3^n (4/3)^n}{\sqrt{n}\, 4^n} = \sum \frac{1}{\sqrt{n}}$$ diverges by p-test

if $x = -4/3$

$$\sum_{n=1}^{\infty} \frac{3^n (-4/3)^n}{\sqrt{n}\, 4^n} = \sum \frac{(-1)^n}{\sqrt{n}}$$ converges by alt-series test

ANS: $\boxed{[-4/3, 4/3)}$

9. Find the value of the geometric series or state that the series diverges and justify your answer. (10 pts.)

$$\sum_{n=2}^{\infty} \frac{3 \cdot 5^n}{3^{2n-1}}$$

$$\sum_{n=1}^{\infty} \frac{3 \cdot 5^n}{3^{2n} \cdot 3^{-1}} = 9 \sum_{n=1}^{\infty} \left(\frac{5^n}{9^n}\right) = 9\left(\frac{1}{1-\frac{5}{9}} - \left(\frac{5}{9}\right)^0\right)$$

$$= 9\left(\frac{1}{\frac{9-5}{9}} - 1\right) - \frac{5}{9}\Big)$$

$$= 9\left(\frac{9}{4} - \frac{4}{4} - \frac{5}{9}\right) = 9 \cdot \left(5/4 - \frac{5}{9}\right) = \boxed{\cancel{45/4}}$$

$$= \frac{45}{4} - 5 = \frac{45}{4} - \frac{20}{4}$$

$$= 25/4$$

10. Which of the following statements are true? Circle all true statements. (2 pts. each)

(a) If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

(b) If $\lim_{n \to \infty} a_n = 0$, then $\sum a_n$ converges.

(c) If $a_n \leq b_n$ and $\sum a_n$ converges, then $\sum b_n$ converges.

(d) $\sum \frac{(-1)^n}{n}$ converges.

(e) $\sum \frac{1}{n!}$ converges.