Math 1552
Summer 2023
Exam 3 Practice

Time limit: 75 Minutes

Name (Print): Key

Canvas email: ____________

Teaching Assistant/Section: ____________

GT ID:

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Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- $\boxed{\textbf{Place a box}}$ around your final answer for full credit.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 6 pages of questions.

1. (~~2~~ 3 points) Which of the following statements are true? *You do not have to show your work.*

true	false	
●	○	The alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n+4}$ diverges.
●	○	The radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{n!}(x-2)^n$ is $R=\infty$.
●	○	The value of e^{-1} is within an error of $\frac{1}{24}$ to $1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}$.

2. (~~4~~ 5 points) State the Taylor series at $x=0$ for the function and give the interval of convergence. You do not have to show work for this problem. *Hint: this is a common Taylor series.*

$f(x)=\ln(1+x)$

$$\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}}$$

$$\boxed{(-1,1)}$$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \quad \text{if } |x|<1$$

$$\ln(1+x)=\int\frac{1}{1+x}\,dx=\int\sum_{n=0}^{\infty}(-1)^n x^n\,dx$$

$$=\sum_{n=0}^{\infty}(-1)^n\int x^n\,dx$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

↑
defined if
$1+x>0$
so $x>-1$

3. (8 points) Find the Taylor series at $x = 0$ for the given function. If you use a common Taylor series state the series you are using.

$$f(x) = x^2 e^{-x}$$

Common

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

So

$$x^2 e^{-x} = x^2 \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n$$

$$= x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2}}$$

4. (8 points) Determine if the given series converges or diverges. Show your work and clearly justify your answer for credit. *Hint: try a comparison test. Hint:* $\ln n \le \sqrt{n}$ *if* $n \ge 4$.

$$\sum_{n=4}^{\infty} \frac{3}{\sqrt{n}\ln n}$$

direct comparison

Since $\ln n \le \sqrt{n}$ if $n \ge 3$

$$\frac{1}{\sqrt{n}\cdot \ln n} \ge \frac{1}{\sqrt{n}\sqrt{n}} = \frac{1}{n} = b_n$$

Now $\sum b_n$ diverges by p-test w/ $p = 1 \le 1$.

So since $\sum a_n \ge \sum b_n$ by direct comparison

$\boxed{\sum a_n \text{ also diverges}}$

Check hint?

want to show

$\ln n \le \sqrt{n}$ if $n \ge 4$

$\ln 4 \overset{?}{\le} 2$ ✓

Now

$(\ln n)' = \frac{1}{n} \le \frac{1}{2\sqrt{n}} = (\sqrt{n})'$

$\Rightarrow \frac{\sqrt{n}}{n} \le \frac{1}{2}$ $(n \ge 4)$

$\Rightarrow \frac{1}{\sqrt{n}} \le \frac{1}{2}$?

$\Rightarrow 2 \le \sqrt{n}$ $n \ge 4$ ✓

Nice!

5. (8 points) Determine if the given series converges or diverges. Show your work and clearly justify your answer for credit.

$$\sum_{n=1}^{\infty} \frac{e^n}{n^e}$$

Ratio test

$a_{n+1} = \dfrac{e^{n+1}}{(n+1)^e}$

$a_n = \dfrac{e^n}{n^e}$

So $\dfrac{a_{n+1}}{a_n} = \dfrac{e^{\cancel{n+1}}}{(n+1)^e} \cdot \dfrac{n^e}{\cancel{e^n}} = e \cdot \left(\dfrac{n}{n+1}\right)^e$

and since $\dfrac{n}{n+1} \to 1$ as $n \to \infty$

$$\lim_{n\to\infty} e\left(\frac{n}{n+1}\right)^e = e \cdot (1)^e = e = L$$

Since $L > 1$, the series

$\boxed{\sum a_n \text{ diverges}}$

6. (8 points) Consider the function $f(x) = \sqrt{x}$.

(a) Find the Taylor polynomial $P_3(x)$ of degree $N = 3$ expanded at $x = 1$ (in powers of $x - 1$).

@ $x=1$

$f(x) = \sqrt{x} = x^{1/2}$ $\quad f(1) = 1$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(1) = 1/2$

$f''(x) = \frac{-1}{4}x^{-3/2}$ $\quad f''(1) = \frac{-1}{4}$

$f^{(3)}(x) = \frac{3}{8}x^{-5/2}$ $\quad f^{(3)}(1) = \frac{3}{8}$

So $\sqrt{x} \approx f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \frac{f^{(3)}(1)}{3!}$

$$\boxed{\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{3}{48}(x-1)^3}$$

(b) Use the formula below to find the maximum error of using $P_3(2)$, the Taylor polynomial evaluated at $x = 2$, to approximate $f(2) = \sqrt{2}$.

$$|R_N(x)| \leq \max |f^{(N+1)}(c)| \frac{|x-a|^{N+1}}{(N+1)!}$$

Need to bound $|f^{(4)}(x)|$ over $[1, 2]$.

$f^{(4)}(x) = \frac{-15}{16}x^{-7/2}$

So $|f^{(4)}(x)| = \frac{15}{16 \cdot x^{7/2}} \leq \frac{15}{16}$ for every x in $[1,2]$

Therefore

$$|R_3(x)| \leq \frac{15}{16} \cdot \frac{|2-1|^4}{4!} = \frac{\overset{5}{\cancel{15}}}{16 \cdot 4 \cdot \cancel{3} \cdot 2 \cdot 1} = \boxed{\frac{5}{128}}$$

$$\frac{n!}{(n+1)!} = \frac{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{(n+1)\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$= \frac{1}{n+1}$$

7. (10 points) Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$.

(a) Find the interval of convergence of the power series $f(x)$.
Hint: don't forget to test the endpoints separately.

Ratio test

$$\boxed{(-\infty, \infty)}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{\cancel{n+1}} x^{\cancel{n+1}}}{(n+1)!} \cdot \frac{n!}{\cancel{2^n}\,\cancel{x^n}} = 2x \cdot \frac{n!}{(n+1)!} = \frac{2x}{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 \quad \text{for any } x \text{ in } (-\infty, \infty)$$

(b) Find a power series for $f'(x)$.

$$f'(x) = \left(\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n\right)'$$

$$= \sum_{n=\cancel{0}\,1}^{\infty} \frac{2^n}{n!} (x^n)' \quad \text{(constant term drops off)}$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{n!} \cdot n x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{(n-1)!} x^{n-1}$$

$$\frac{n}{n!} = \frac{\cancel{n}}{\cancel{n}(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = \frac{1}{(n-1)!}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{2^n}{(n-1)!} x^{n-1}}$$

This page may be used for scratch work. Please indicate clearly on the problem if you would like your work on this page to be graded. Loose scrap paper is not permitted. This page must **not** ***be detached*** *from your exam booklet at any time.*