Math 1552, Integral Calculus

Test 3 Review, Sections 10.3-10.9

1. Terminology review: complete the following statements.

(a) A geometric series has the general form _____________.

The series converges when _____________ and diverges when _____________.

(b) A p-series has the general form _____________. The series converges when _____________ and diverges when _____________. To show these results, we can use the _____________ test.

(c) The harmonic series has the form _____________, and it _____________.

(d) If you want to show a series converges, compare it to a _____________ series that also converges. If you want to show a series diverges, compare it to a _____________ series that also diverges.

(e) If the direct comparison test does not have the correct inequality, you can instead use the _____________ _____________ test. In this test, if the limit is a _____________ number (not equal to _____________), then both series converge or both series diverge.

(f) In the ratio and root tests, the series will _____________ if the limit is less than 1 and _____________ if the limit is greater than 1. If the limit equals 1, then the test is _____________.

(g) If $\sum_k a_k$ is an alternating series, then it converges _____________ if $\sum_k |a_k|$ converges. It converges _____________ if $\sum_k |a_k|$ diverges and (i) the limit of the terms is _____________ and (ii) the sequence of terms is _____________.

(h) If an alternating series converges, we can estimate the sum by adding the first n terms. Stopping after n terms will give us an error at most equal to the magnitute of the _____________ term in the sequence.

(i) If $\lim_{n\to\infty} a_n = 0$, then what, if anything, do we know about the series $\sum_n a_n$? _____________

(j) A power series has the general form: _____________. To find the radius of convergence R, use either the _____________ or _____________ test. The series converges _____________ when $|x - c| < R$. To find the interval of convergence, don't forget to check the _____________.

(k) A Taylor polynomial has the general form: ______________. The Taylor polynomial is the n^{th} ______________ ______________ of the Taylor series with general form: ______________.

(l) The Taylor remainder theorem says that $|R_n| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$, where M represents the maximum value of the ______________ derivative of f on the interval between x and a. The remainder term decreases when n ______________ or when x is ______________ to a.

(m) A MacLaurin Series is a Taylor series centered at ______________.

(n) Complete the formulas for the common MacLaurin series.

$$e^x = \sum_{k=0}^{\infty}$$

$$\ln(1+x) = \sum_{k=0}^{\infty}$$

$$\sin(x) = \sum_{k=0}^{\infty}$$

$$\cos(x) = \sum_{k=0}^{\infty}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty}$$

(o) Fill in the formulas for the derivatives and anti-derivatives of a power series.

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k x^k\right] =$$

$$\int_0^x \left[\sum_{k=0}^{\infty} a_k t^k\right] dt =$$

Problems from Recitation Worksheets

2. Determine if each of the following statements is always true or sometimes false.

(a) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+1}}$ is a p-series with $p = \frac{3}{2}$.

(b) $\sum_{k=1}^{\infty} \frac{1}{k(\ln k)^p}$ converges when $p > 1$.

(c) To show a series $\sum_k a_k$ converges by the Basic Comparison Test, we should find a smaller series $\sum_k b_k$ that also converges.

(d) A limit of 0 or ∞ from the Limit Comparison Test may not give us a conclusive answer as to whether our series converges or diverges.

(e) To determine whether $\sum_{k=3}^{\infty} \frac{k}{k^3-10}$ converges or diverges, use the Basic Comparison Test with $\sum_{k=3}^{\infty} \frac{1}{k^2}$.

(f) If $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$, then the series $\sum_k a_k$ converges.

(g) We should use the root test if all of the terms are raised to the k^{th} power.

(h) We can use the root test to show that the p-series $\sum_k \frac{1}{\sqrt{k}}$ diverges.

(i) The ratio test would be inconclusive for the series $\sum_k \frac{k}{k^3+1}$.

(j) $(2k)! = 2k!$

(k) If an alternating series converges absolutely, then it also converges conditionally.

(l) If $\sum_k |a_k|$ converges, then the alternating series $\sum_k a_k$ also converges.

(m) If $\sum_k a_k$ is an alternating series and $\{|a_k|\}$ is a decreasing sequence, then $\sum_k a_k$ converges.

(n) If $\sum_k a_k$ is an alternating series and $\sum_k |a_k|$ diverges, then $\sum_k a_k$ cannot converge absolutely.

(o) If $\sum_k a_k$ is an alternating series and $\lim_{k\to\infty} |a_k| \neq 0$, then $\sum_k a_k$ diverges.

3. Determine whether the following series converge or diverge. Justify your answers using any of the tests we have discussed in class. Make sure that you (1) name the test and state the conditions needed for the test you are using, (2) show work for the test that requires some math, and (3) state a conclusion that explains why the test shows convergence or divergence.

(a) $\sum_{k=1}^{\infty} \frac{e^k}{4+e^{2k}}$

(b) $\sum_{k=1}^{\infty} \left(1 - \frac{3}{k}\right)^k$

(c) $\sum_{k=1}^{\infty} k \tan\left(\frac{1}{k}\right)$

(d) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$

(e) $\sum_{k=1}^{\infty} \frac{3^{2k}}{8^k-3}$

(f) $\sum_{k=1}^{\infty} \frac{k+2}{\sqrt{k^5+4}}$

(g) $\sum_{k=1}^{\infty} \frac{k+3}{\sqrt{k^2+1}}$

(h) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ...$

(i) $\sum_{k=1}^{\infty} \frac{\ln k}{k^4}$

(j) $\sum_{k=1}^{\infty} \frac{(2k)^k}{k!}$

(k) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{2k^2}$

(l) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{4^n 2^n n!}$

4. Suppose $r > 0$. Find the values of r, if any, for which $\sum_{k=1}^{\infty} \frac{r^k}{k^r}$ converges.

5. Determine whether the following alternating series converge absolutely, converge conditionally, or diverge. Justify your answers using the tests we discussed in class.

(a) $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{3k}{\sqrt{k^3+4}}$

(b) $\sum_{k=2}^{\infty} (-1)^{k} \frac{k}{k^4-1}$

(c) $\sum_{k=0}^{\infty} (-1)^{k} \frac{k}{5^k+2^k}$

(d) $\sum_{k=2}^{\infty} (-1)^{k} \frac{1}{k \ln k \sqrt{\ln \ln k}}$

6. Find the radius and interval of convergence of the following power series:

(a) $\sum_{k=2}^{\infty} \left(\frac{k}{k-1}\right) \frac{(x+2)^k}{2^k}$

(b) $\sum_{n=1}^{\infty} \frac{(3x+2)^n}{\sqrt{n}}$

7. Find a power series representation for the function $f(x) = \frac{x}{4+x^4}$. For what values of x does the series converge?

8. Find the third degree Taylor polynomial of the function $f(x) = \tan^{-1}(x)$ in powers of $x - 1$.

9. Use a Taylor polynomial to estimate the value of $\sqrt{e}$ with an error of at most 0.01. HINT: Choose $a = 0$ and use the fact that $e < 3$.

10. For what values of x can we replace $\cos x$ with $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ within an error range of no more that 0.001?

11. Use the MacLaurin series for $f(x) = \frac{1}{1-x}$ to find a power series representation of the function

$$g(x) = \frac{x}{(1-x)^3}.$$

HINT: You will need to differentiate.

12. Find $f^{(7)}(0)$ for the function $f(x) = x\sin(x^2)$.

13. Find a power series (i.e., MacLaurin series) representation for the following functions. When is your series valid?

(a) $f(x) = \frac{3x}{2+4x}$

(b) $g(x) = xe^{-x}$

14. Find a MacLaurin series for the function $f(x) = \tan^{-1} x$.

15. Find the sum of the series:

$$\frac{\pi}{2} - \frac{\pi^3}{8 \cdot 3!} + \frac{\pi^5}{32 \cdot 5!} + ... + (-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} + ...$$

16. Use a MacLaurin series to estimate $\int_0^1 e^{-x^2} dx$ within an error of no more than 0.01.

Additional Test Review Problems

17. Let $\{a_n\}$ and $\{b_n\}$ be a sequences of non-negative terms. Are the following statements *always* true or sometimes false?

(a) If $\lim_{n\to\infty} a_n = L$, then the series $\sum_n a_n = L$.

(b) If $\lim_{n\to\infty} a_n = 0$, then $\{a_n\}$ converges to 0.

(c) If $\lim_{n\to\infty} a_n = 0$, then $\sum_n a_n$ converges.

(d) If $\sum_n a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(e) If $\sum_n a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

(f) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_n a_n$ diverges.

(g) If $\lim_{n\to\infty} a_n \neq 0$, then $\{a_n\}$ diverges.

(h) If $\int_1^\infty f(x)dx = L$, where $0 < L < \infty$, then $\sum_n f(n) = L$.

(i) If $\sum_n b_n$ converges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also converges.

(j) If $\sum_n b_n$ diverges and $a_n > b_n$ for all $n \geq 1$, then $\sum_n a_n$ also diverges.

(k) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_n b_n$ diverges, then $\sum_n a_n$ also diverges.

18. Determine if each statement below is always true or sometimes false.

(a) If $p > 1$ the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges conditionally.

(b) The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is conditionally convergent.

(c) If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ also converges.

(d) If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, then $\sum_{n=1}^{\infty} \sin(a_n)$ also converges.

(e) The series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for any $p > 1$.

(f) The series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges for any $p \leq 1$.

(g) Let $\{a_n\}$ be a positive sequence and $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ be a finite number. If a series $\sum_{n=1}^{\infty} a_n$ converges, then $L < 1$.

(h) Let $\{a_n\}$ be a positive sequence and $L = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$ be a finite number. If a series $\sum_{n=1}^{\infty} a_n$ diverges, then $L \geq 1$.

(i) If the radius of convergence of a power series is 0, then the power series diverges everywhere.

(j) If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges in $(-1, 1)$, then its radius of convergence is 1.

(k) Every Taylor series is a power series.

(l) The fifth degree Taylor polynomial for $\cos(x)$ about $x = 0$ is $1 - \frac{x^2}{2} + \frac{x^4}{4!}$.

(m) For any Taylor polynomial, the error in the approximation is no more than the magnitude of the $(n+1)^{st}$ term.

19. Let $f(x) = \int_0^x t\sin(t^3)dt$. Use a MacLaurin series to find $f^{(11)}(0)$.

20. (a) Estimate $\cos\left(\frac{\pi}{12}\right)$ using a fourth-degree Taylor polynomial.

(b) Find a MacLaurin series for the function $g(x) = \int_0^x \frac{\sin(t/2)}{2t}dt$.

21. Determine if the alternating series converges absolutely or converges conditionally. Justify your answer fully by: (1) name the test and state the conditions needed for the

test you are using, (2) show work for the test that requires some math, and (3) state a conclusion that explains why the test shows convergence or divergence.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty}(-5)^{-n}$

22. Determine whether the given series converges or diverges. Make sure that you (1) name the test and state the conditions needed for the test you are using, (2) show work for the test that requires some math, and (3) state a conclusion that explains why the test shows convergence or divergence.

(a) $\sum_{n=2}^{\infty} \frac{n+1}{\sqrt{n^3-2}}$

(b) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ (Hint: Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.)

(c) $\sum_{n=3}^{\infty} \frac{10}{n \ \ln n \ \ln(\ln n)}$

(d) $\sum_{n=1}^{\infty} ne^{-n}$

(e) $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}$

(f) $\sum_{n=1}^{\infty} \left(\frac{\sin n}{1+n}\right)^n$

(g) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{2 \cdot 3^{n-1}}$

(h) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

23. Find the radius and interval of convergence of each power series below.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n (2x-4)^n}{n^2}$

(b) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n x^n$

Answers

2. (b), (d), (g), (i), (l), (n), and (o) are true
3. (a), (d), (f), (h), (i), (k), and (l) converge
4. converges when $0 < r < 1$
5. (a) and (d) converge conditionally, (b) and (c) converge absolutely
6. (a) $R = 2$, $I.C. = (-4, 0)$, (b) $R = \frac{1}{3}$, $I.C. = \left[-1, -\frac{1}{3}\right)$
7. $\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{4^{k+1}}$, $|x| < 4^{1/4}$
8. $\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$
9. 1.6458
10. $x \in (-0.9467, 0.9467)$
11. $\frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-1}$
12. -840
13. (a) $3\sum_{k=0}^{\infty}(-1)^k 2^{k-1} x^{k+1}$, $|x| < \frac{1}{2}$ (b) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k!}$, $x \in \Re$
14. $\sum_{k=0}^{\infty}(-1)^k \cdot \frac{x^{2k+1}}{2k+1}$, $|x| < 1$
15. 1
16. ≈ 0.743
17. Statements (b), (d), (f), and (j) are true.
18. (b), (c), (d), (e), (f), (h), (k), and (l) are true
19. $-\frac{10!}{6}$
20. (a) 0.966, (b) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2^{2k+2}(2k+1)!(2k+1)}$
21. (a) converges conditionally; (b) converges absolutely
22. (b), (d), (e), (f), (g), and (h) converge
23. (a) $R = \frac{1}{2}$, $I.C. = \left[\frac{3}{2}, \frac{5}{2}\right]$; (b) $R = 1$, $I.C. = (-1, 1)$