

MATH

1552

Chapter 10

sequences/series    integral test

# Math 1552

## Section 10.2 Infinite Series



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## Infinite Series

An *infinite series* is a *sum* of infinite terms:

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \dots + a_n + \dots$$

Ex 1: 
$$\sum_{n=1}^{\infty} \frac{3}{10^n}$$

Ex 2: 
$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

## Review Question:

Which of the following sequences converge?

- (A)  $\left\{ \frac{2n+1}{1-3n} \right\}$   
 (B)  $\{(-1)^n\}$   
 (C)  $\left\{ \frac{2^n}{n!} \right\}$   
 (D)  $\left\{ \left(1 + \frac{4}{n}\right)^n \right\}$

Which of these series do you think converges?

- (A)  $\sum_{n=1}^{\infty} \frac{1}{n}$   
 (B)  $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$   
 (C)  $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$   
 (D) None of these

Ex.  $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)}$

Hint: How would you deal  
with  $\int_1^{\infty} \frac{1}{(x+1)(x+3)} dx$ ?

Ex.  $\sum_{k=2}^{\infty} \ln\left(\frac{n}{n+1}\right)$

```
clc
k=5;
N=10;
A=0;
for i=1:k
    N=10^i;
    for n=2:N
        A=A+log(n/(n+1));
    end
end
```

N	A
10	18.00
100	-1.70
1000	108.00
10000	-5.63
100000	1000.00
1000000	-11.84
10000000	10000.00
100000000	-28.36
1000000000	100000.00
10000000000	-31.18

Ex.

$$\sum_{n=2}^{\infty} \frac{5^{n-1} + 3 \cdot 2^{3n}}{9^n}$$

## THM: Divergence test.

If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then

$$\sum_{n=1}^{\infty} a_n \text{ surely diverges .}$$

Ex. Determine if  $a_n \rightarrow 0$  or not to see if the divergence test applies in each case.

(a)  $\sum_{k=1}^{\infty} \left(1 + \frac{3}{k}\right)^k$

(b)  $\sum_{k=2}^{\infty} \frac{3k}{5k-7}$

(c)  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+6}}$

Q: If  $\lim_{n \rightarrow \infty} a_n = 0$ , then do we know for sure that  $\sum_{n=1}^{\infty} a_n$  converges?

Which statement is always true?

If  $\lim_{n \rightarrow \infty} a_n = 0$ , then :

- A. The series converges.
- B. The sequence converges.
- C. The sequence of partial sums converges.
- D. The series diverges.



Section 10.2: 7, 29, 45, 49, 59, 63, 79, 95  
(extra practice: 65, 67, 71, 77, 84, 94, 96)Finding  $n$ th Partial Sums

In Exercises 1–6, find a formula for the  $n$ th partial sum of each series and use it to find the series' sum if the series converges.

1.  $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

2.  $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

3.  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1} \frac{1}{2^{n-1}} + \cdots$

4.  $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1} 2^{n-1} + \cdots$

5.  $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

6.  $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

## Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7.  $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

8.  $\sum_{n=2}^{\infty} \frac{1}{4^n}$

9.  $\sum_{n=1}^{\infty} \left(1 - \frac{7}{4^n}\right)$

10.  $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

11.  $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$

12.  $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

13.  $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

14.  $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

In Exercises 15–22, determine if the geometric series converges or diverges. If a series converges, find its sum.

15.  $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$

16.  $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

17.  $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$

18.  $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$

19.  $1 - \left(\frac{2}{e}\right) + \left(\frac{2}{e}\right)^2 - \left(\frac{2}{e}\right)^3 + \left(\frac{2}{e}\right)^4 - \cdots$

20.  $\left(\frac{1}{3}\right)^{-2} - \left(\frac{1}{3}\right)^{-1} + 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 - \cdots$

21.  $1 + \left(\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^4 + \left(\frac{10}{9}\right)^6 + \left(\frac{10}{9}\right)^8 + \cdots$

22.  $\frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \frac{243}{32} + \frac{729}{64} - \cdots$

## Repeating Decimals

Express each of the numbers in Exercises 23–30 as the ratio of two integers.

23.  $0.\overline{23} = 0.23\ 23\ 23\ \dots$

24.  $0.\overline{234} = 0.234\ 234\ 234\ \dots$

25.  $0.\overline{7} = 0.7777\ \dots$

26.  $0.\overline{d} = 0.d\ d\ d\ \dots$ , where  $d$  is a digit

27.  $0.0\overline{6} = 0.06666\ \dots$

28.  $1.\overline{414} = 1.414\ 414\ 414\ \dots$

29.  $1.24\overline{123} = 1.24\ 123\ 123\ 123\ \dots$

30.  $3.\overline{142857} = 3.142857\ 142857\ \dots$

Using the  $n$ th-Term Test

In Exercises 31–38, use the  $n$ th-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

31.  $\sum_{n=1}^{\infty} \frac{n}{n+10}$

32.  $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

33.  $\sum_{n=0}^{\infty} \frac{1}{n+4}$

34.  $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

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35.  $\sum_{n=1}^{\infty} \cos \frac{1}{n}$

36.  $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$

37.  $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

38.  $\sum_{n=0}^{\infty} \cos n\pi$

## Telescoping Series

In Exercises 39–44, find a formula for the  $n$ th partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

39.  $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

40.  $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

41.  $\sum_{n=1}^{\infty} (\ln \sqrt{n+1} - \ln \sqrt{n})$

42.  $\sum_{n=1}^{\infty} (\tan(n) - \tan(n-1))$

43.  $\sum_{n=1}^{\infty} \left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n+2}\right)\right)$

44.  $\sum_{n=1}^{\infty} (\sqrt{n+4} - \sqrt{n+3})$

Find the sum of each series in Exercises 45–52.

45.  $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

46.  $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

47.  $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

48.  $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

49.  $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

50.  $\sum_{n=1}^{\infty} \left(\frac{1}{2^n/n} - \frac{1}{2^{n+1}/(n+1)}\right)$

51.  $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

52.  $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

69.  $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

70.  $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

71.  $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

72.  $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{n\pi}}$

73.  $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} - \frac{n+2}{n+3}\right)$

74.  $\sum_{n=2}^{\infty} \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$

75.  $\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right)$

76.  $\sum_{n=0}^{\infty} (\ln(4e^n - 1) - \ln(2e^n + 1))$

Geometric Series with a Variable  $x$ 

In each of the geometric series in Exercises 77–80, write out the first few terms of the series to find  $a$  and  $r$ , and find the sum of the series. Then express the inequality  $|r| < 1$  in terms of  $x$  and find the values of  $x$  for which the inequality holds and the series converges.

77.  $\sum_{n=0}^{\infty} (-1)^n x^n$

78.  $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

79.  $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

80.  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2} \left(\frac{1}{3 + \sin x}\right)^n$

In Exercises 81–86, find the values of  $x$  for which the given geometric series converges. Also, find the sum of the series (as a function of  $x$ ) for those values of  $x$ .

81.  $\sum_{n=0}^{\infty} 2^n x^n$

82.  $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$

83.  $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

84.  $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

## Convergence or Divergence

Which series in Exercises 53–76 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

53. 
$$\sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^n$$

54. 
$$\sum_{n=0}^{\infty} (\sqrt{2})^n$$

55. 
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$$

56. 
$$\sum_{n=1}^{\infty} (-1)^{n+1} n$$

57. 
$$\sum_{n=0}^{\infty} \cos \left( \frac{n\pi}{2} \right)$$

58. 
$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$$

59. 
$$\sum_{n=0}^{\infty} e^{-2n}$$

60. 
$$\sum_{n=1}^{\infty} \ln \frac{1}{3^n}$$

61. 
$$\sum_{n=1}^{\infty} \frac{2}{10^n}$$

62. 
$$\sum_{n=0}^{\infty} x^n, \quad |x| > 1$$

63. 
$$\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$$

64. 
$$\sum_{n=1}^{\infty} \left( 1 - \frac{1}{n} \right)^n$$

65. 
$$\sum_{n=0}^{\infty} \frac{n!}{1000^n}$$

66. 
$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

67. 
$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$$

68. 
$$\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$$

92. Find convergent geometric series  $A = \sum a_n$  and  $B = \sum b_n$  that illustrate the fact that  $\sum a_n b_n$  may converge without being equal to  $AB$ .

93. Show by example that  $\sum (a_n/b_n)$  may converge to something other than  $A/B$  even when  $A = \sum a_n$ ,  $B = \sum b_n \neq 0$ , and no  $b_n$  equals 0.

94. If  $\sum a_n$  converges and  $a_n > 0$  for all  $n$ , can anything be said about  $\sum (1/a_n)$ ? Give reasons for your answer.

95. What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.

96. If  $\sum a_n$  converges and  $\sum b_n$  diverges, can anything be said about their term-by-term sum  $\sum (a_n + b_n)$ ? Give reasons for your answer.

97. Make up a geometric series  $\sum ar^{n-1}$  that converges to the number 5 if

a.  $a = 2$                       b.  $a = 13/2$ .

98. Find the value of  $b$  for which

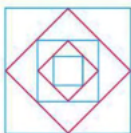
$$1 + e^b + e^{2b} + e^{3b} + \dots = 9.$$

99. For what values of  $r$  does the infinite series

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \dots$$

converge? Find the sum of the series when it converges.

100. The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of  $4 \text{ m}^2$ . Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



101. **Drug dosage** A patient takes a 300 mg tablet for the control of high blood pressure every morning at the same time. The concentration of the drug in the patient's system decays exponentially at a constant hourly rate of  $k = 0.12$ .

a. How many milligrams of the drug are in the patient's system just before the second tablet is taken? Just before the third tablet is taken?

b. In the long run, after taking the medication for at least six months, what quantity of drug is in the patient's body just before taking the next regularly scheduled morning tablet?

102. Show that the error  $(L - s_n)$  obtained by replacing a convergent geometric series with one of its partial sums  $s_n$  is  $ar^n/(1 - r)$ .

85. 
$$\sum_{n=0}^{\infty} \sin^n x$$

86. 
$$\sum_{n=0}^{\infty} (\ln x)^n$$

## Theory and Examples

87. The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+4)}.$$

Write it as a sum beginning with (a)  $n = -2$ , (b)  $n = 0$ , (c)  $n = 5$ .

88. The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{5}{(n+1)(n+2)}.$$

Write it as a sum beginning with (a)  $n = -1$ , (b)  $n = 3$ , (c)  $n = 20$ .

89. Make up an infinite series of nonzero terms whose sum is

a. 1    b.  $-3$     c. 0.

90. (Continuation of Exercise 89.) Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

91. Show by example that  $\sum (a_n/b_n)$  may diverge even though  $\sum a_n$  and  $\sum b_n$  converge and no  $b_n$  equals 0.

103. **The Cantor set** To construct this set, we begin with the closed interval  $[0, 1]$ . From that interval, remove the middle open interval  $(1/3, 2/3)$ , leaving the two closed intervals  $[0, 1/3]$  and  $[2/3, 1]$ . At the second step we remove the open middle third interval from each of those remaining. From  $[0, 1/3]$  we remove the open interval  $(1/9, 2/9)$ , and from  $[2/3, 1]$  we remove  $(7/9, 8/9)$ , leaving behind the four closed intervals  $[0, 1/9]$ ,  $[2/9, 1/3]$ ,  $[2/3, 7/9]$ , and  $[8/9, 1]$ . At the next step, we remove the middle open third interval from each closed interval left behind, so  $(1/27, 2/27)$  is removed from  $[0, 1/9]$ , leaving the closed intervals  $[0, 1/27]$  and  $[2/27, 1/9]$ ;  $(7/27, 8/27)$  is removed from  $[2/9, 1/3]$ , leaving behind  $[2/9, 7/27]$  and  $[8/27, 1/3]$ , and so forth. We continue this process repeatedly without stopping, at each step removing the open third interval from every closed interval remaining behind from the preceding step. The numbers remaining in the interval  $[0, 1]$ , after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845–1918). The set has some interesting properties.

a. The Cantor set contains infinitely many numbers in  $[0, 1]$ . List 12 numbers that belong to the Cantor set.

b. Show, by summing an appropriate geometric series, that the total length of all the open middle third intervals that have been removed from  $[0, 1]$  is equal to 1.

104. **Helga von Koch's snowflake curve** Helga von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

a. Find the length  $L_n$  of the  $n$ th curve  $C_n$  and show that

$$\lim_{n \rightarrow \infty} L_n = \infty.$$

b. Find the area  $A_n$  of the region enclosed by  $C_n$  and show that

$$\lim_{n \rightarrow \infty} A_n = (8/5)A_1.$$



105. The largest circle in the accompanying figure has radius 1. Consider the sequence of circles of maximum area inscribed in semi-circles of diminishing size. What is the sum of the areas of all of the circles?





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Section 10.3

The Integral test

$$\int_1^{\infty} f(x) dx \quad \text{vs.} \quad \sum_{k=1}^{\infty} f(k)$$



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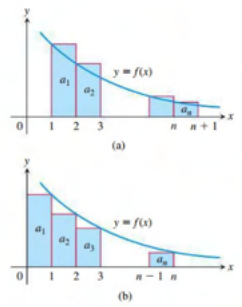


FIGURE 10.12 Subject to the conditions of the Integral Test, the series  $\sum_{n=1}^{\infty} a_n$  and the integral  $\int_1^{\infty} f(x) dx$  both converge or both diverge.

Thm:  
**Integral TEST**  
 IF  $y=f(x)$  is positive and decreasing for all  $x \geq 1$ , THEN  
 The series

$$\sum_{k=1}^{\infty} f(k) \text{ converges}$$

if and only if the integral

$$\int_1^{\infty} f(x) dx \text{ converges}$$

Ex. Use the integral test to determine if (a)  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  converges or diverges.

(b)  $\sum_{k=1}^{\infty} \frac{1}{k}$

So For example

$\int_1^{\infty} \frac{1}{x^2} dx$  tells us about the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

And  $\int_1^{\infty} \frac{1}{x} dx$  tells us about

$$\sum_{k=1}^{\infty} \frac{1}{k} \quad (\text{etc.})$$

Ex

$$\sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$$

Ex.

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1}$$

Ex.

$$\sum_{k=1}^{\infty} k e^{-k^2}$$

The  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if  $p > 1$ , diverges if  $p \leq 1$ .

## EXERCISES 10.3

## Applying the Integral Test

Use the Integral Test to determine if the series in Exercises 1–12 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2.  $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$

3.  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

4.  $\sum_{n=1}^{\infty} \frac{1}{n+4}$

5.  $\sum_{n=1}^{\infty} e^{-2n}$

6.  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

7.  $\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$

8.  $\sum_{n=2}^{\infty} \frac{\ln(n^2)}{n}$

9.  $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

10.  $\sum_{n=2}^{\infty} \frac{n-4}{n^2 - 2n + 1}$

11.  $\sum_{n=1}^{\infty} \frac{7}{\sqrt{n+4}}$

12.  $\sum_{n=2}^{\infty} \frac{1}{5n + 10\sqrt{n}}$

## Determining Convergence or Divergence

Which of the series in Exercises 13–46 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

13.  $\sum_{n=1}^{\infty} \frac{1}{10^n}$

14.  $\sum_{n=1}^{\infty} e^{-n}$

15.  $\sum_{n=1}^{\infty} \frac{n}{n+1}$

16.  $\sum_{n=1}^{\infty} \frac{5}{n+1}$

17.  $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

18.  $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$

19.  $\sum_{n=1}^{\infty} -\frac{1}{8^n}$

20.  $\sum_{n=1}^{\infty} \frac{-8}{n}$

21.  $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

22.  $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

23.  $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

24.  $\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3}$

25.  $\sum_{n=0}^{\infty} \frac{-2}{n+1}$

26.  $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

27.  $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

28.  $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

29.  $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$

30.  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

31.  $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$

32.  $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$

33.  $\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$

34.  $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)}$

35.  $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

36.  $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$

37.  $\sum_{n=1}^{\infty} \frac{e^n}{1 + e^{2n}}$

38.  $\sum_{n=1}^{\infty} \frac{2}{1 + e^n}$

39.  $\sum_{n=1}^{\infty} \frac{e^n}{10 + e^n}$

40.  $\sum_{n=1}^{\infty} \frac{e^n}{(10 + e^n)^2}$

41.  $\sum_{n=2}^{\infty} \frac{\sqrt{n+2} - \sqrt{n+1}}{\sqrt{n+1}\sqrt{n+2}}$

42.  $\sum_{n=3}^{\infty} \frac{7}{\sqrt{n+1}\ln\sqrt{n+1}}$

43.  $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1 + n^2}$

44.  $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

45.  $\sum_{n=1}^{\infty} \operatorname{sech} n$

46.  $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

partial sums just grow too slowly. To see what we mean, suppose you had started with  $s_1 = 1$  the day the universe was formed, 13 billion years ago, and added a new term every second. About how large would the partial sum  $s_n$  be today, assuming a 365-day year?

50. Are there any values of  $x$  for which  $\sum_{n=1}^{\infty} (1/nx)$  converges? Give reasons for your answer.

51. Is it true that if  $\sum_{n=1}^{\infty} a_n$  is a divergent series of positive numbers, then there is also a divergent series  $\sum_{n=1}^{\infty} b_n$  of positive numbers with  $b_n < a_n$  for every  $n$ ? Is there a “smallest” divergent series of positive numbers? Give reasons for your answers.

52. (Continuation of Exercise 51.) Is there a “largest” convergent series of positive numbers? Explain.

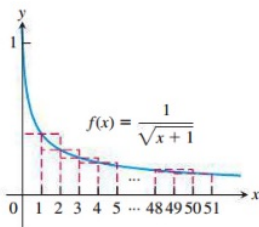
53.  $\sum_{n=1}^{\infty} (1/\sqrt{n+1})$  diverges

a. Use the accompanying graph to show that the partial sum

$$s_{50} = \sum_{n=1}^{50} (1/\sqrt{n+1})$$

$$s_{50} < \int_1^{51} \frac{1}{\sqrt{x+1}} dx < \int_0^{50} \frac{1}{\sqrt{x+1}} dx.$$

Conclude that  $11.5 < s_{50} < 12.3$ .



b. What should  $n$  be in order that the partial sum

$$s_n = \sum_{i=1}^n (1/\sqrt{i+1})$$

$$\text{satisfy } s_n > 1000?$$

54.  $\sum_{n=1}^{\infty} (1/n^4)$  converges

a. Use the accompanying graph to find an upper bound for the error if  $s_{30} = \sum_{n=1}^{30} (1/n^4)$  is used to estimate the value of  $\sum_{n=1}^{\infty} (1/n^4)$ .