WEEK 8

Series Review & Power series

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

Tentative Course Schedule

Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral
2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution
3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution
5	Jun 12 Section 8.5: Partial fractions Section 4.5: L'Hopital's	Jun 13 WS 8.4 WS 8.5	Jun 14 Section 8.8: Improper Integrals	Jun 15 WS 8.5, 4.5 Quiz #3 (8.4-8.5)	Jun 16 Section 10.1: Sequences
6	Jun 19 **NO CLASS** Juneteenth	Jun 20 WS 8.8 WS 10.1	Jun 21 Section 10.2: Infinite Series	Jun 22 WS 10.1 cont. Quiz #4 (4.5, 8.8, 10.1)	Jun 23 Section 10.3: Integral Test
7	Jun 26 Section 10.4: Comparison Tests	Jun 27 WS 10.2 WS 10.3	Jun 28 Section 10.5: Ratio and Root Tests Review for Test 2	Jun 29 **Test #2 (8.4-8.5, 4.5, 8.8, 10.1-10.3)**	Jun 30 Section 10.5: cont. Section 10.6: Alternating Series
8	Jul 3 **NO CLASS** Independence Day	Jul 4 **NO CLASS** Student Recess	Jul 5 Section 10.6: cont. Section 10.7: Power series	Jul 6 WS 10.4 WS 10.5 Quiz #5 (10.4-10.5)	Jul 7 Section 10.7, cont.
9	Jul 10 Sections 10.8-10.9: Taylor polynomials and series	Jul 11 WS 10.6 WS 10.7	Jul 12 Sections 10.8-10.9, cont.	Jul 13 WS 10.8-10.9 Quiz #6 (10.6-10.8)	Jul 14 Sections 10.8-10.9, cont.
10	Jul 17 Sections 10.8-10.9, cont.	Jul 18 WS 10.8-10.9 (3 versions)	Jul 19 Sections 10.8-10.9, cont.	Jul 20 **Test #3 (10.4-10.9)**	Jul 21 Section 6.1: Volumes by Disks
11	Jul 24 Section 6.1: Volumes by Cylindrical Shells Final Review	Jul 25 WS 6.1-6.2 *Last day for MML homework*	Jul 26 Reading Day	Jul 27	Jul 28 **FINAL EXAM** **11:20 AM – 2:10 PM**
12	Jul 31	Aug 1	Aug 2	Aug 3	Aug 4

Final Exam # Questions

≈ 2x length of midterm

100% more questions

$\boxed{\text{2hr 50 min}}$

113% more time.

SERIES REVIEW BOXES.

Geometric Series Formula

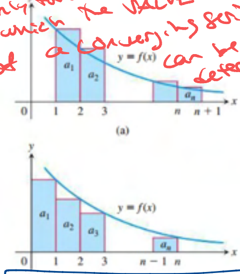
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

if $|r| < 1$

~~telescoping series~~

only two

of which the VALUE of a converging series can be determined



THM: Divergence test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ surely diverges.

The *p*-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p > 1$, diverges if $p \leq 1$.

The Integral test

$$\int_1^{\infty} f(x)\,dx \quad \text{vs.} \quad \sum_{k=1}^{\infty} f(k)$$

THEOREM 10—Direct Comparison Test

Let Σa_n and Σb_n be two series with $0 \leq a_n \leq b_n$ for all n. Then

1. If Σb_n converges, then Σa_n also converges.
2. If Σa_n diverges, then Σb_n also diverges.

THEOREM 11—Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then Σa_n and Σb_n both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and Σb_n converges, then Σa_n converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and Σb_n diverges, then Σa_n diverges.

Ratio Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

(a) If $L < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

(b) If $L > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $L = 1$, then the test is *INCONCLUSIVE!!!*

Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $R = \lim_{n\to\infty} \sqrt[n]{a_n}$.

(a) If $R < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

(b) If $R > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $R = 1$, then the test is *INCONCLUSIVE!!!*

Alternating Series Test

Let $\sum_k a_k$ be an alternating series.

(a) If $\sum_k |a_k|$ converges, then the series *converges absolutely*.

(b) If (a) fails, then if:

i) $\{|a_n|\}$ is a decreasing sequence, and

ii) $\lim_{n\to\infty} |a_n| = 0$,

then the series *converges conditionally*.

(c) Otherwise, the series *diverges*.

Estimating the Sum

Let $\sum_k a_k$ be a convergent alternating series with a sum of L.

Then: $|s_n - L| < |a_{n+1}|$.

Ex. Converge or Diverge?

State the test you are using along with any necessary conditions for the test to work.

(a) $$\sum_{n=1}^{\infty} \frac{3n}{2n^2-1}$$

① Name the test you are using.

Soln. $\boxed{\text{Try limit comparison.}}$

$$a_n = \frac{3n}{2n^2-1} \qquad b_n = \frac{3}{2n}$$

② define a_n & b_n

Side work

$$\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

Harmonic series diverges by p-series w/ $p=1$ (or integral test)

Next $$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} = \frac{\cancel{3}n}{2n^2-1} \cdot \frac{2n}{\cancel{3}} = \frac{2n^2}{2n^2-1}$$

and $$\lim_{n\to\infty} \frac{a_n}{b_n} = \boxed{1 = c}$$

③ compute limit & tell me the condition on limit value

Since $\boxed{c>0,}$ by the limit comparison test both $\sum a_n$ & $\sum b_n$ diverge or converge

④ Conclusion. Summary

Since $\sum b_n$ diverges (Harmonic series)

So too $\boxed{\sum a_n \text{ diverges}}$

⑤ answer.

(b) $$\sum_{n=1}^{\infty} (-1)^n \frac{3n}{2n^2-1}$$

Converges absolutely? conditionally? Diverges?

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n}{2n^2-1}$ Converges absolutely? conditionally? Diverges?

In part (a) we saw $\sum a_n$ diverges.

Soln. To check for conditional convergence we need to check

$$\lim_{n\to\infty} a_n = 0$$

So $\displaystyle\lim_{n\to\infty} \frac{3n}{2n^2-1} \overset{?}{=} \lim_{n\to\infty} \frac{3}{4n} = 0.$ Since $a_n \to 0$ $\sum(-1)^n a_n$ converges by the alt. series test.

L'Hop "$\frac{\infty}{\infty}$"

$\boxed{\text{The series } \sum(-1)^n \frac{3n}{2n^2-1} \text{ Converges conditionally}}$

EXERCISES 10.2

Section 10.2: 7, 29, 45, 49, 59, 63, 79, 95
(extra practice: 65, 67, 71, 77, 84, 94, 96)

Finding nth Partial Sums

In Exercises 1–6, find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$
2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$
3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$
4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}2^{n-1} + \cdots$
5. $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$
6. $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$
8. $\sum_{n=2}^{\infty} \frac{1}{4^n}$
9. $\sum_{n=1}^{\infty} \left(1 - \frac{7}{4^n}\right)$
10. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$
11. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$
12. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$
13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$
14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

In Exercises 15–22, determine if the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$
16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$
17. $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$
18. $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$
19. $1 - \left(\frac{2}{e}\right) + \left(\frac{2}{e}\right)^2 - \left(\frac{2}{e}\right)^3 + \left(\frac{2}{e}\right)^4 - \cdots$
20. $\left(\frac{1}{3}\right)^{-2} - \left(\frac{1}{3}\right)^{-1} + 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 - \cdots$
21. $1 + \left(\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^4 + \left(\frac{10}{9}\right)^6 + \left(\frac{10}{9}\right)^8 + \cdots$
22. $\frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \frac{243}{32} + \frac{729}{64} - \cdots$

Repeating Decimals

Express each of the numbers in Exercises 23–30 as the ratio of two integers.

23. $0.\overline{23} = 0.23\ 23\ 23\ldots$
24. $0.\overline{234} = 0.234\ 234\ 234\ldots$
25. $0.\overline{7} = 0.7777\ldots$
26. $0.\overline{d} = 0.dddd\ldots$, where d is a digit
27. $0.0\overline{6} = 0.06666\ldots$
28. $1.\overline{414} = 1.414\ 414\ 414\ldots$
29. $1.24\overline{123} = 1.24\ 123\ 123\ 123\ldots$
30. $3.\overline{142857} = 3.142857\ 142857\ldots$

Using the nth-Term Test

In Exercises 31–38, use the nth-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

31. $\sum_{n=1}^{\infty} \frac{n}{n+10}$
32. $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$
33. $\sum_{n=0}^{\infty} \frac{1}{n+4}$
34. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

35. $\sum_{n=1}^{\infty} \cos\frac{1}{n}$
36. $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$
37. $\sum_{n=1}^{\infty} \ln\frac{1}{n}$
38. $\sum_{n=0}^{\infty} \cos n\pi$

Telescoping Series

In Exercises 39–44, find a formula for the nth partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

39. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$
40. $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$
41. $\sum_{n=1}^{\infty} \left(\ln\sqrt{n+1} - \ln\sqrt{n}\right)$
42. $\sum_{n=1}^{\infty} \left(\tan(n) - \tan(n-1)\right)$
43. $\sum_{n=1}^{\infty} \left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n+2}\right)\right)$
44. $\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3}\right)$

Find the sum of each series in Exercises 45–52.

45. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$
46. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$
47. $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$
48. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$
49. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$
50. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}}\right)$
51. $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$
52. $\sum_{n=1}^{\infty} \left(\tan^{-1}(n) - \tan^{-1}(n+1)\right)$

69. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$
70. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$
71. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$
72. $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$
73. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} - \frac{n+2}{n+3}\right)$
74. $\sum_{n=2}^{\infty} \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$
75. $\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right)$
76. $\sum_{n=0}^{\infty} \left(\ln(4e^n - 1) - \ln(2e^n + 1)\right)$

Geometric Series with a Variable x

In each of the geometric series in Exercises 77–80, write out the first few terms of the series to find a and r, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

77. $\sum_{n=0}^{\infty} (-1)^n x^n$
78. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$
79. $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$
80. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

In Exercises 81–86, find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

81. $\sum_{n=0}^{\infty} 2^n x^n$
82. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$
83. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$
84. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

Convergence or Divergence

Which series in Exercises 53–76 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

53. $\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ **54.** $\sum_{n=0}^{\infty}(\sqrt{2})^n$

55. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$ **56.** $\sum_{n=1}^{\infty}(-1)^{n+1}n$

57. $\sum_{n=0}^{\infty}\cos\left(\frac{n\pi}{2}\right)$ **58.** $\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$

59. $\sum_{n=0}^{\infty}e^{-2n}$ **60.** $\sum_{n=1}^{\infty}\ln\frac{1}{3^n}$

61. $\sum_{n=1}^{\infty}\frac{2}{10^n}$ **62.** $\sum_{n=0}^{\infty}\frac{1}{x^n}, \quad |x| > 1$

63. $\sum_{n=0}^{\infty}\frac{2^n - 1}{3^n}$ **64.** $\sum_{n=1}^{\infty}\left(1 - \frac{1}{n}\right)^n$

65. $\sum_{n=0}^{\infty}\frac{n!}{1000^n}$ **66.** $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

67. $\sum_{n=1}^{\infty}\frac{2^n + 3^n}{4^n}$ **68.** $\sum_{n=1}^{\infty}\frac{2^n + 4^n}{3^n + 4^n}$

85. $\sum_{n=0}^{\infty}\sin^n x$ **86.** $\sum_{n=0}^{\infty}(\ln x)^n$

Theory and Examples

87. The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} \quad \text{and} \quad \sum_{n=-1}^{\infty}\frac{1}{(n+3)(n+4)}.$$

Write it as a sum beginning with **(a)** $n = -2$, **(b)** $n = 0$, **(c)** $n = 5$.

88. The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty}\frac{5}{n(n+1)} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{5}{(n+1)(n+2)}.$$

Write it as a sum beginning with **(a)** $n = -1$, **(b)** $n = 3$, **(c)** $n = 20$.

89. Make up an infinite series of nonzero terms whose sum is

a. 1 **b.** -3 **c.** 0.

90. (*Continuation of Exercise 89.*) Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

91. Show by example that $\sum(a_n/b_n)$ may diverge even though $\sum a_n$ and $\sum b_n$ converge and no b_n equals 0.

92. Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to AB.

93. Show by example that $\sum(a_n/b_n)$ may converge to something other than A/B even when $A = \sum a_n$, $B = \sum b_n \neq 0$, and no b_n equals 0.

94. If $\sum a_n$ converges and $a_n > 0$ for all n, can anything be said about $\sum(1/a_n)$? Give reasons for your answer.

95. What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.

96. If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum(a_n + b_n)$? Give reasons for your answer.

97. Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5 if

a. $a = 2$ **b.** $a = 13/2$.

98. Find the value of b for which

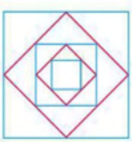
$$1 + e^b + e^{2b} + e^{3b} + \cdots = 9.$$

99. For what values of r does the infinite series

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$$

converge? Find the sum of the series when it converges.

100. The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m². Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



101. Drug dosage A patient takes a 300 mg tablet for the control of high blood pressure every morning at the same time. The concentration of the drug in the patient's system decays exponentially at a constant hourly rate of $k = 0.12$.

a. How many milligrams of the drug are in the patient's system just before the second tablet is taken? Just before the third tablet is taken?

b. In the long run, after taking the medication for at least six months, what quantity of drug is in the patient's body just before taking the next regularly scheduled morning tablet?

102. Show that the error $(L - s_n)$ obtained by replacing a convergent geometric series with one of its partial sums s_n is $ar^n/(1 - r)$.

103. The Cantor set To construct this set, we begin with the closed interval $[0, 1]$. From that interval, remove the middle open interval $(1/3, 2/3)$, leaving the two closed intervals $[0, 1/3]$ and $[2/3, 1]$. At the second step we remove the open middle third interval from each of those remaining. From $[0, 1/3]$ we remove the open interval $(1/9, 2/9)$, and from $[2/3, 1]$ we remove $(7/9, 8/9)$, leaving behind the four closed intervals $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$, and $[8/9, 1]$. At the next step, we remove the middle open third interval from each closed interval left behind, so $(1/27, 2/27)$ is removed from $[0, 1/9]$, leaving the closed intervals $[0, 1/27]$ and $[2/27, 1/9]$; $(7/27, 8/27)$ is removed from $[2/9, 1/3]$, leaving behind $[2/9, 7/27]$ and $[8/27, 1/3]$, and so forth. We continue this process repeatedly without stopping, at each step removing the open third interval from every closed interval remaining behind from the preceding step. The numbers remaining in the interval $[0, 1]$, after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845–1918). The set has some interesting properties.

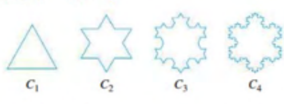
a. The Cantor set contains infinitely many numbers in $[0, 1]$. List 12 numbers that belong to the Cantor set.

b. Show, by summing an appropriate geometric series, that the total length of all the open middle third intervals that have been removed from $[0, 1]$ is equal to 1.

104. Helga von Koch's snowflake curve Helga von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

a. Find the length L_n of the nth curve C_n and show that $\lim_{n\to\infty} L_n = \infty$.

b. Find the area A_n of the region enclosed by C_n and show that $\lim_{n\to\infty} A_n = (8/5)A_1$.



105. The largest circle in the accompanying figure has radius 1. Consider the sequence of circles of maximum area inscribed in semicircles of diminishing size. What is the sum of the areas of all of the circles?

Section 10.4: 5, 9, 17, 21, 25, 34, 41
(extra practice: 13, 18, 23, 31, 39, 51, 58)

EXERCISES 10.4

Direct Comparison Test

In Exercises 1–8, use the Direct Comparison Test to determine if each series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$
2. $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$
3. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$
4. $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$
5. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$
6. $\sum_{n=1}^{\infty} \frac{1}{n3^n}$
7. $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$
8. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

Limit Comparison Test

In Exercises 9–16, use the Limit Comparison Test to determine if each series converges or diverges.

9. $\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$
   (*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/n^2)$)
10. $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$
   (*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/\sqrt{n})$)
11. $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$
12. $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$
13. $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$
14. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$
15. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
   (*Hint:* Limit Comparison with $\sum_{n=2}^{\infty}(1/n)$)
16. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$
   (*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/n^2)$)

Determining Convergence or Divergence

Which of the series in Exercises 17–56 converge, and which diverge? Use any method, and give reasons for your answers.

17. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$
18. $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$
19. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$
20. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$
21. $\sum_{n=1}^{\infty} \frac{2n}{3n-1}$
22. $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$
23. $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$
24. $\sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$
25. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$
26. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$
27. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$
28. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$
29. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$
30. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$
31. $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$
32. $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$
33. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$
34. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$
35. $\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$
36. $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$
37. $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$
38. $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$
39. $\sum_{n=1}^{\infty} \frac{n+1}{n^2+3n}\cdot\frac{1}{5n}$
40. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$
41. $\sum_{n=1}^{\infty} \frac{2^n-n}{n2^n}$
42. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\,e^n}$
43. $\sum_{n=2}^{\infty} \frac{1}{n!}$
   (*Hint:* First show that $(1/n!) \le (1/n(n-1))$ for $n \ge 2$.)
44. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$
45. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$
46. $\sum_{n=1}^{\infty} \tan\frac{1}{n}$
47. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.1}}$
48. $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}}$
49. $\sum_{n=1}^{\infty} \frac{\coth n}{n^2}$
50. $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$
51. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$
52. $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$
53. $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$
54. $\sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$
55. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$
56. $\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n}$

Theory and Examples

57. Prove **(a)** Part 2 and **(b)** Part 3 of the Limit Comparison Test.
58. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty}(a_n/n)$? Explain.
59. Suppose that $a_n > 0$ and $b_n > 0$ for $n \ge N$ (N an integer). If $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum a_n$ converges, can anything be said about $\sum b_n$? Give reasons for your answer.
60. Prove that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ converges.
61. Suppose that $a_n > 0$ and $\lim_{n\to\infty} a_n = \infty$. Prove that $\sum a_n$ diverges.
62. Suppose that $a_n > 0$ and $\lim_{n\to\infty} n^2 a_n = 0$. Prove that $\sum a_n$ converges.

Section 10.5: 7, 13, 15, 17, 18, 31, 34, 57, 63, 67
(extra practice: 7, 17, 18, 19, 21, 25, 37, 42, 59)

EXERCISES 10.5

Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine if each series converges absolutely or diverges.

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2. $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$

3. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$

4. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$

5. $\sum_{n=1}^{\infty} \frac{n^4}{(-4)^n}$

6. $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

7. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2(n+2)!}{n!\, 3^{2n}}$

8. $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Using the Root Test

In Exercises 9–16, use the Root Test to determine if each series converges absolutely or diverges.

9. $\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$

10. $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

11. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

12. $\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

13. $\sum_{n=1}^{\infty} \frac{-8}{(3+(1/n))^{2n}}$

14. $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

15. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$

(*Hint:* $\lim_{n\to\infty} (1 + x/n)^n = e^x$)

16. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1+n}}$

Determining Convergence or Divergence

In Exercises 17–46, use any method to determine if the series converges or diverges. Give reasons for your answer.

17. $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

18. $\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n}$

19. $\sum_{n=1}^{\infty} n!(-e)^{-n}$

20. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

21. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

22. $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

23. $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$

24. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

25. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{3}{n}\right)^n$

26. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$

27. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

28. $\sum_{n=1}^{\infty} \frac{(-\ln n)^n}{n^n}$

29. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$

30. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

31. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

32. $\sum_{n=1}^{\infty} \frac{n \ln n}{(-2)^n}$

33. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

34. $\sum_{n=1}^{\infty} e^{-n}(n^3)$

35. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

36. $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

37. $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

38. $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$

39. $\sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n}$

40. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{(n/2)}}$

41. $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$

42. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$

43. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

44. $\sum_{n=1}^{\infty} \frac{(2n+3)(2^n+3)}{3^n+2}$

45. $\sum_{n=3}^{\infty} \frac{2^n}{n^2}$

46. $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}}$

55. $a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt[n]{a_n}$

56. $a_1 = \frac{1}{2}, \quad a_{n+1} = (a_n)^{n+1}$

Convergence or Divergence

Which of the series in Exercises 57–64 converge, and which diverge? Give reasons for your answers.

57. $\sum_{n=1}^{\infty} \frac{2^n n! n!}{(2n)!}$

58. $\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!}{n!(n+1)!(n+2)!}$

59. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

60. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^n}{n^{(n^2)}}$

61. $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$

62. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

63. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{4^n 2^n n!}$

64. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{[2 \cdot 4 \cdot \cdots \cdot (2n)](3^n+1)}$

65. Assume that b_n is a sequence of positive numbers converging to $4/5$. Determine if the following series converge or diverge.

a. $\sum_{n=1}^{\infty} (b_n)^{1/n}$

b. $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n (b_n)$

c. $\sum_{n=1}^{\infty} (b_n)^n$

d. $\sum_{n=1}^{\infty} \frac{1000^n}{n! + b_n}$

66. Assume that b_n is a sequence of positive numbers converging to $1/3$. Determine if the following series converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{b_{n+1} b_n}{n 4^n}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!\, b_1^2 b_2^2 \cdots b_n^2}$

Theory and Examples

67. Neither the Ratio Test nor the Root Test helps with p-series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and show that both tests fail to provide information about convergence.

MML/Pearson §10.5

$5! = 5\cdot4\cdot3\cdot2\cdot1 = 5\cdot4!$
$4! = 4\cdot3\cdot2\cdot1$

Ratio test.

#1 $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$$\boxed{\frac{a_{n+1}}{a_n}} = \boxed{2^{n+1}}\,\frac{}{(n+1)!}\cdot\frac{n!}{\boxed{2^n}} = \frac{\cancel{2^n}\cdot 2}{\cancel{2^n}}\cdot\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}$$

$2^n\cdot2^1$ $\quad \frac{1}{a_n}$ $\quad a_{n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{2}{n+1} \xrightarrow[n\to\infty]{} 0 = L$$

Using ratio test. ① Name the test

$a_n = \frac{2^n}{n!}$ write a_n ② & a_{n+1}

$a_{n+1} \overset{?}{=} \frac{\cancel{2^n+1}}{\cancel{n!+1}}$

$a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

③ form ratio & take limit

Since $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 = L$

and $L<1$ by the

ratio test ...

④ justification summary

$\boxed{\sum a_n \text{ converges}}$

⑤ answer

#4 $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n+1}}{n\,3^{n-1}}$ Ratio test

$\displaystyle\lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+0} = \lim 1 = 1.$ L'Hop "$\frac{\infty}{\infty}$"

Soln using ratio test ①

$a_n = \frac{2^{n+1}}{n\,3^{n-1}}$ ②

$a_{n+1} = \frac{2^{(n+1)+1}}{(n+1)\,3^{(n+1)-1}} = \frac{2^{n+2}}{(n+1)3^n}$

So $\displaystyle\lim\frac{a_{n+1}}{a_n} = \lim\frac{2}{3}\cdot\frac{n}{n+1}$

③ $= \frac{2}{3}\cdot 1 = \frac{2}{3} = L$

Since $L<1$ by ratio test. ④

$\boxed{\sum a_n \text{ converges}}$ ⑤

① Name the test
② define/write appropriate terms
③ Do THE WORK
④ Summary
⑤ Answer

So,

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+2}}{(n+1)3^n}\cdot\frac{n\cdot3^{n-1}}{2^{n+1}} = \frac{\cancel{2^n}\cdot\cancel{2^2}^{\,2}\cdot n\cdot\cancel{3^n}\cdot\frac{1}{3}}{(n+1)\,\cancel{3^n}\cdot\cancel{2^n}\cdot\cancel{2}} = \frac{2}{3}\cdot\frac{n}{n+1}$$

③

Ex. § 10.4 textbook #12.

$$\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$$

Q: Converge or diverge?

Soln. Let's try direct comparison. (1)

Guess converges

$$a_n = \frac{2^n}{3+4^n} \le \frac{2^n}{4^n} = b_n \text{ \& } \sum b_n \text{ converges}$$

a_n & b_n (2)

(3)

this one.

b/c

$$\sum \frac{2^n}{4^n} = \sum \left(\frac{2}{4}\right)^n$$

geometric series w/ $r = 1/2 < 1$

Guess diverges

$$a_n = \frac{2^n}{3+4^n} \ge \quad \text{\& } \sum b_n \text{ diverges.}$$

Since $a_n \le b_n$ & $\sum b_n$ converges

by direct comparison (4)

$\boxed{\sum a_n \text{ converges}}$ (5)

$\int \frac{1}{x^2+1}\,dx$??

??

$$\sum \frac{1}{n^2+1} \le \sum \frac{1}{n^2}$$

$\int \frac{1}{x^2}\,dx$

1. Name the test
2. define/write appropriate terms
3. Do THE WORK
4. Summary
5. Answer.

Why $a_n \le b_n$

Why $a_n \ge b_n$?

$$\frac{1}{2} + \frac{1}{2^2+1} + \frac{1}{3^2+1} \cdots \le 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\sum a_n \le \sum b_n = 4$$

vs.

$$\sum a_n \ge \sum b_n$$

Section 10.6: 6, 19, 21, 23, 25, 27, 34, 35, 37, 47
(extra practice: 20, 22, 27, 31, 91)

EXERCISES 10.6

Determining Convergence or Divergence

In Exercises 1–14, determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$
3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n3^n}$
4. $\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$
5. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$
6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+5}{n^2+4}$
7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$
8. $\sum_{n=1}^{\infty}(-1)^{n}\frac{10^n}{(n+1)!}$
9. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

root test.

10. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$
11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$
12. $\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$
13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$
14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$

Absolute and Conditional Convergence

Which of the series in Exercises 15–48 converge absolutely, which converge, and which diverge? Give reasons for your answers.

15. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$
16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$
17. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$
18. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$
19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$
20. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$
21. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$
22. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$
23. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$
24. $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$
25. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$
26. $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt[n]{10})$
27. $\sum_{n=1}^{\infty}(-1)^{n}n^2(2/3)^n$
28. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$
29. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\tan^{-1}n}{n^2+1}$
30. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n-\ln n}$
31. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$
32. $\sum_{n=1}^{\infty}(-5)^{-n}$
33. $\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$
34. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$
35. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$
36. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$
37. $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^n}{(2n)^n}$
38. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n!)^2}{(2n)!}$

Error Estimation

In Exercises 49–52, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

49. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$
50. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n}$
51. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n}$ As you will see in Section 10.7, the sum is ln (1.01).
52. $\frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n,\quad 0<t<1$

In Exercises 53–56, determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

53. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n^2+3}$
54. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$
55. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(n+3\sqrt{n})^3}$
56. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\ln(\ln(n+2))}$

In Exercises 57–82, use any method to determine whether the series converges or diverges. Give reasons for your answer.

57. $\sum_{n=1}^{\infty}\frac{3^n}{n^n}$
58. $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$
59. $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$
60. $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)$
61. $\sum_{n=0}^{\infty}(-1)^{n}\frac{(n+2)!}{(2n)!}$
62. $\sum_{n=2}^{\infty}\frac{(3n)!}{(n!)^3}$
63. $\sum_{n=1}^{\infty}n^{-2/\sqrt{5}}$
64. $\sum_{n=2}^{\infty}\frac{3}{10+n^{4/3}}$
65. $\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^{n^2}$
66. $\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^n$
67. $\sum_{n=1}^{\infty}\frac{n-2}{n^2+3n}\left(-\frac{2}{3}\right)^n$
68. $\sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^n$

Math 1552
Section 10.7

Power Series

↳ a_n sequence

↳ $\sum a_n$ series

↳ $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

8	Jul 3 **NO CLASS** Independence Day	Jul 4 **NO CLASS** Student Recess	Jul 5 Section 10.6: cont. Section 10.7: Power series	Jul 6 WS 10.4 WS 10.5 Quiz #5 (10.4-10.5)	Jul 7 Section 10.7, cont.
9	Jul 10 Sections 10.8-10.9: Taylor polynomials and series	Jul 11 WS 10.6 WS 10.7	Jul 12 Sections 10.8-10.9, cont.	Jul 13 WS 10.8-10.9 Quiz #6 (10.6-10.8)	Jul 14 Sections 10.8-10.9, cont.
10	Jul 17 Sections 10.8-10.9, cont.	Jul 18 WS 10.8-10.9 (3 versions)	Jul 19 Sections 10.8-10.9, cont.	Jul 20 **Test #3 (10.4-10.9)**	Jul 21 Section 6.1: Volumes by Disks
11	Jul 24 Section 6.1: Volumes by Cylindrical Shells Final Review	Jul 25 WS 6.1-6.2 *Last day for MML homework*	Jul 26 Reading Day	Jul 27	Jul 28 **FINAL EXAM** **11:20 AM – 2:10 PM**
12	Jul 31	Aug 1	Aug 2	Aug 3	Aug 4

Review Question:
The series: $\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{k^2+1}}$

A. Converges absolutely
B. Converges conditionally
C. Diverges

Q: For which x-values does this power series converge?

$$\sum_{k=1}^{\infty}\frac{x^k}{k}$$

$x=7$?

$\sum_{k=1}^{\infty}\frac{7^k}{k}$ Series converge or diverge?

$\lim_{k\to\infty}\frac{7^k}{k} = +\infty$ DNE

So by divergence test $\sum\frac{7^k}{k}$ diverges.

$x=0$?

$\sum_{k=1}^{\infty}\frac{0^k}{k} = \frac{0^1}{1}+\frac{0^2}{2}+\frac{0^3}{3}+\dots = 0$

series converges $\sum\frac{0^k}{k}$ ✓

Power Series

A *power series* is an *infinite polynomial* and a *function* of *x*:

Power series in x: $f(x)=\sum_{k=0}^{\infty}a_k x^k$

Power series in $x-c$: $f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k$

$x=1$. $\sum_{k=1}^{\infty}\frac{1^k}{k} = \sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-series w/ $p=1$.

$x=1/2$ $\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)^k}{k} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\cdot\frac{1}{k} = \sum_{k=1}^{\infty}\frac{1}{2^k\cdot k}$

$x=-1$?,

root test

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{2^k \cdot k} \text{ converges by root test w/ } L = 1/2 < 1}$$

$b_k = k^{1/k} \to 1.$

$\lim k^{1/k} = \boxed{1}$ $\ln(1) = 0$

$a_k = \frac{1}{2^k \cdot k}$

$$(a_k)^{1/k} = \left(\frac{1}{2^k \cdot k}\right)^{1/k}$$

$$= \frac{1}{(2^k)^{1/k} \cdot k^{1/k}}$$

$$= \frac{1}{2 \cdot k^{1/k}}$$

$$\lim (a_k)^{1/k} = \frac{1}{2 \cdot 1} = \boxed{\frac{1}{2}} = L$$

$$\ln(b_k) = \ln(k^{1/k}) = \frac{1}{k} \cdot \ln k \to 0$$

L'Hop ∞/∞

$$\lim_{k\to\infty} \frac{\ln k}{k} = \lim_{k\to\infty} \frac{1/k}{1}$$

$$= \lim \frac{1}{k} = 0$$

$$b_k = k^{1/k^2}$$

no bueno

$$\infty^0$$

$$\lim \Big(\underbrace{1 + \frac{1}{n}}_{1}\Big)^{n \to \infty} = e$$

$$\ln(b_k) = \frac{1}{k^2} \ln k$$

$$\lim \frac{\ln k}{k^2} = \lim \frac{1/k}{2k}$$

$$= \lim \frac{1}{k} \cdot \frac{1}{2k}$$

$$\sum_{k=1}^{\infty} \frac{x^k}{k} \leftarrow$$

$x = 7$ or 3 or 1 DIVERGES

$x = 0$ or $1/2$ CONVERGES

$x = -1$ converges.

$x = -2$ diverges $\sum \frac{(-2)^k}{k}$.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = \frac{-1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$= \underbrace{\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}}$$

$\sum \frac{1}{k}$ diverges

converges by alt. series test since $\frac{1}{k} \to 0$.

x in $[-1, 1)$ or $-1 \le x < 1$

What is the condition that x must satisfy so that $\sum \frac{x^k}{k}$ converges?

Guess $|x| \le 1$??

Soln. Try RATIO TEST $a_k = \frac{x^k}{k}$

Guess $\boxed{-1 \le x < 1}$??

$$\frac{a_{k+1}}{a_k} = \underbrace{\frac{x^{k+1}}{k+1}}_{a_{k+1}} \cdot \underbrace{\frac{k}{x^k}}_{1/a_k} = \frac{\cancel{x^k} \cdot x}{\cancel{x^k}} \cdot \frac{k}{k+1}$$

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} x \cdot \frac{k}{k+1}$$

$$= x \cdot 1 = \boxed{x = L}$$

x in $(-1, 1)$ ✓

If $|x| < 1$ then power series converges

$x \in (-\infty, -1) \cup (1, \infty)$

If $|x| > 1$ then power series diverges

$x = \pm 1$??

If $|x| = 1$ inconclusive

check one at a time

Ex. What x-values make this power series converge to a number?

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}\, x^n}{3^n}$$

$|x|<1$

Soln
ratio test

$$\frac{a_{n+1}}{a_n} = \underbrace{\frac{\sqrt{n+1}\cdot x^{n+1}}{3^{n+1}}}_{a_{n+1}} \cdot \underbrace{\frac{3^n}{\sqrt{n}\cdot x^n}}_{\frac{1}{a_n}} = \frac{x}{3}\cdot\frac{\sqrt{n+1}}{\sqrt{n}} = \frac{x}{3}\sqrt{\frac{n+1}{n}}$$

$$\lim_{n\to\infty} \frac{x}{3}\cdot\sqrt{\frac{n+1}{n}} = \frac{x}{3}\cdot 1 = \frac{x}{3} = L$$

So if

$\left|\frac{x}{3}\right| < 1$ convergence of $\sum \frac{\sqrt{n}\,x^n}{3^n}$

if $\left|\frac{x}{3}\right| > 1$ divergence of $\sum \frac{\sqrt{n}\,x^n}{3^n}$

if $\left|\frac{x}{3}\right| = 1$ test inconclusive.

Solve $\left|\frac{x}{3}\right| < 1$

so $-1 < \frac{x}{3} < 1$

$\underbrace{-3 < x < 3}$ convergence.

test $x=3$ and $x=-3$ separately.

When $x=3$

$1+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\cdots$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}\cdot 3^n}{3^n} = \sum_{n=1}^{\infty} \sqrt{n}$$

diverges by divergence test since $\lim \sqrt{n} \neq 0$.

ANS Power series converges if x is in $(-3,3)$ or $-3<x<3$

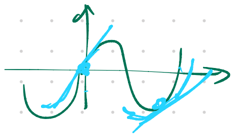
When $x=-3$

$-1+\sqrt{2}-\sqrt{3}+\sqrt{4}-\sqrt{5}+\sqrt{6}+\cdots$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}\,(-3)^n}{3^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n}\,(-1)^n\cdot \cancel{3^n}}{\cancel{3^n}} = \sum_{n=1}^{\infty} \sqrt{n}\,(-1)^n$$

diverges by divergence test $\lim (-1)^n\sqrt{n} \neq 0$

$-6 < x+1 < 6$??



$y = \sin x$

$R=6$

$$\sum_{k=1}^{\infty}\frac{(x+1)^k}{6^k}$$

Radius and I.C.

To find the radius of convergence of a power series in **standard form**, use the ratio or root test to find:

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{ or } L=\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

(i) If $L=0$, then $R=\infty$ and I.C. is all real numbers.

(ii) If $L=\infty$, then $R=0$ and I.C. is just $x=c$.

(iii) If L is positive and finite, then $R=1/L$, and the series converges for $|x-c|<R$. You must also check the endpoints.

ratio test

$$\frac{a_{k+1}}{a_k} = \frac{(x+1)^{k+1}}{6^{k+1}} \cdot \frac{6^k}{(x+1)^k} = \frac{\cancel{(x+1)^k}\cdot(x+1)\cdot\cancel{6^k}}{\cancel{6^k}\cdot 6\cdot\cancel{(x+1)^k}}$$

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{x+1}{6} = \frac{x+1}{6} = L$$

ANS $\boxed{x \text{ is in } (-7,5)}$

$-7<x<5$

Set

$$\left|\frac{x+1}{6}\right| < 1 \Rightarrow -1 < \frac{x+1}{6} < 1$$

$$\Rightarrow -6 < x+1 < 6$$

$$\Rightarrow -6-1 < x < 6-1$$

$$\Rightarrow -7 < x < 5$$

If $\left|\frac{x+1}{6}\right| = 1$

test inconclusive.

Happen when $x=-7$ or $x=5$

Test individually!

If $x=5$
$$\sum_{k=1}^{\infty}\frac{(5+1)^k}{6^k} = \sum_{k=1}^{\infty}\frac{6^k}{6^k} = \sum_{k=1}^{\infty}1 \quad \text{diverges}$$

$-1+1-1+1-1+1\ldots$

$x=-7$
$$\sum_{k=1}^{\infty}\frac{(-7+1)^k}{6^k} = \sum_{k=1}^{\infty}\frac{(-6)^k}{6^k} = \sum_{k=1}^{\infty}\frac{(-1)^k\cdot\cancel{6^k}}{\cancel{6^k}} = \sum_{k=1}^{\infty}(-1)^k$$

diverges.

$$\sum_{k=1}^{\infty}\frac{(4-3x)^k}{\sqrt{2k+5}}$$

$$\sum_{k=1}^{\infty} \frac{(4-3x)^k}{\sqrt{2k+5}}$$

Find the interval of convergence
the center of the interval
& radius of convergence.

$(k+1)+5 = k+6$

Soln.

ratio test

$$\frac{a_{k+1}}{a_k} = \frac{(4-3x)^{k+1}}{\sqrt{2(k+1)+5}} \cdot \frac{\sqrt{2k+5}}{(4-3x)^k} = \frac{\cancel{(4-3x)^k} \cdot (4-3x) \cdot \sqrt{2k+5}}{\sqrt{2k+7} \cdot \cancel{(4-3x)^k}}$$

??

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} (4-3x) \sqrt{\frac{2k+5}{2k+7}}$$

$$= (4-3x) \cdot \sqrt{1}$$

$$= 4-3x = L$$

Next

$$|4-3x| < 1$$

$$\Rightarrow -1 < 4-3x < 1$$

$$\Rightarrow -5 < -3x < -3$$

$$\Rightarrow \boxed{\frac{5}{3} > x > 1}$$

$1 < x < 5/3$ & test $x=1$, $x=5/3$ separately.

$$\sum_{k=1}^{\infty} \frac{(x+1)^k}{6^k}$$

Find x-values where power series converges
(in other words the interval of convergence)

Soln root test

$$a_k = \frac{(x+1)^k}{6^k} \qquad (a_k)^{1/k} = \left(\frac{(x+1)^k}{6^k}\right)^{1/k} = \frac{x+1}{6} = L$$

$$\left|\frac{x+1}{6}\right| < 1.$$

Check $x=-7$ & $x=5$ separately. $\Bigg\}$ (see previous page.)

$$\Rightarrow -1 < \frac{x+1}{6} < 1$$

$$\Rightarrow -6 < x+1 < 6$$

$$\Rightarrow -7 < x < 5$$

$(-7,5)$ interval of convergence

$$\sum_{k=0}^{\infty} \frac{2^k}{k^3}(x-1)^k$$

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely,

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. If R is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If R is finite, the series diverges for $|x - a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Differentiation and Integration

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$\int\left(\sum_{k=0}^{\infty} a_k x^k\right)dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$$

The radius and interval of convergence are preserved under differentiation and Integration.

Example 2:

(a) Find a power series for the function $f(x) = \ln(1+x)$.
For what values of x is this formula valid?

(b) Find a power series representation for $f(x) = \dfrac{1}{(1-x)^2}$.

Section 10.7: 3, 9, 11, 15, 17, 27, 31, 41, 43, 50 (extra practice: 13, 23)

EXERCISES 10.7

Intervals of Convergence

In Exercises 1–36, **(a)** find the series' radius and interval of convergence. For what values of x does the series converge **(b)** absolutely, **(c)** conditionally?

1. $\sum_{n=0}^{\infty} x^n$ **2.** $\sum_{n=0}^{\infty} (x+5)^n$

3. $\sum_{n=0}^{\infty} (-1)^n(4x+1)^n$ **4.** $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

5. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ **6.** $\sum_{n=0}^{\infty} (2x)^n$

7. $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$ **8.** $\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{n}$

9. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$ **10.** $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

11. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ **12.** $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

13. $\sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n}$ **14.** $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

15. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$ **16.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$

17. $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ **18.** $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$

19. $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$ **20.** $\sum_{n=1}^{\infty} \sqrt[n]{n}(2x+5)^n$

21. $\sum_{n=1}^{\infty} (2+(-1)^n)\cdot(x+1)^{n-1}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}(x-2)^n}{3n}$

23. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$ **24.** $\sum_{n=1}^{\infty} (\ln n)x^n$

25. $\sum_{n=1}^{\infty} n^n x^n$ **26.** $\sum_{n=0}^{\infty} n!(x-4)^n$

27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$ **28.** $\sum_{n=0}^{\infty} (-2)^n(n+1)(x-1)^n$

29. $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ Get the information you need about $\sum 1/(n(\ln n)^2)$ from Section 10.3, Exercise 61.

30. $\sum_{n=2}^{\infty} \frac{x^n}{n\ln n}$ Get the information you need about $\sum 1/(n \ln n)$ from Section 10.3, Exercise 60.

31. $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$ **32.** $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

33. $\sum_{n=1}^{\infty} \frac{1}{2\cdot4\cdot6\cdots(2n)}x^n$

34. $\sum_{n=1}^{\infty} \frac{3\cdot5\cdot7\cdots(2n+1)}{n^2\cdot2^n}x^{n+1}$

35. $\sum_{n=1}^{\infty} \frac{1+2+3+\cdots+n}{1^2+2^2+3^2+\cdots+n^2}x^n$

36. $\sum_{n=1}^{\infty} (\sqrt{n+1}-\sqrt{n})(x-3)^n$

In Exercises 37–40, find the series' radius of convergence.

37. $\sum_{n=1}^{\infty} \frac{n!}{3\cdot6\cdot9\cdots3n}x^n$

38. $\sum_{n=1}^{\infty} \left(\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)}\right)^2 x^n$

39. $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n(2n)!}x^n$

40. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

(*Hint:* Apply the Root Test.)

In Exercises 41–48, use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function

45. $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2}-1\right)^n$ **46.** $\sum_{n=0}^{\infty} (\ln x)^n$

47. $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ **48.** $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

Using the Geometric Series

49. In Example 2 we represented the function $f(x) = 2/x$ as a power series about $x = 2$. Use a geometric series to represent $f(x)$ as a power series about $x = 1$, and find its interval of convergence.

50. Use a geometric series to represent each of the given functions as a power series about $x = 0$, and find their intervals of convergence.

a. $f(x) = \frac{5}{3-x}$ **b.** $g(x) = \frac{3}{x-2}$

51. Represent the function $g(x)$ in Exercise 50 as a power series about $x = 5$, and find the interval of convergence.

52. a. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8}{4^{n+2}}x^n.$$

b. Represent the power series in part (a) as a power series about $x = 3$ and identify the interval of convergence of the new series. (Later in the chapter you will understand why the new interval of convergence does not necessarily include all of the numbers in the original interval of convergence.)

Theory and Examples

53. For what values of x does the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

54. If you integrate the series in Exercise 53 term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

55. The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

converges to $\sin x$ for all x.

a. Find the first six terms of a series for $\cos x$. For what values of x should the series converge?

b. By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x.

c. Using the result in part (a) and series multiplication, calculate the first six terms of a series for $2 \sin x \cos x$. Compare your answer with the answer in part (b).

56. The series

Section 10.7: 3, 9, 11, 15, 17, 27, 31, 41, 43, 50 (extra practice: 13, 23)

In Exercises 41–48, use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

41. $\sum_{n=0}^{\infty} 3^n x^n$

42. $\sum_{n=0}^{\infty} (e^x - 4)^n$

43. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$

44. $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

answer with the answer in part (b).

56. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x for all x.

a. Find a series for $(d/dx)e^x$. Do you get the series for e^x? Explain your answer.

b. Find a series for $\int e^x\,dx$. Do you get the series for e^x? Explain your answer.

c. Replace x by $-x$ in the series for e^x to find a series that converges to e^{-x} for all x. Then multiply the series for e^x and e^{-x} to find the first six terms of a series for $e^{-x} \cdot e^x$.

57. The series

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

converges to $\tan x$ for $-\pi/2 < x < \pi/2$.

a. Find the first five terms of the series for $\ln|\sec x|$. For what values of x should the series converge?

b. Find the first five terms of the series for $\sec^2 x$. For what values of x should this series converge?

c. Check your result in part (b) by squaring the series given for $\sec x$ in Exercise 58.

58. The series

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$$

converges to $\sec x$ for $-\pi/2 < x < \pi/2$.

a. Find the first five terms of a power series for the function $\ln|\sec x + \tan x|$. For what values of x should the series converge?

b. Find the first four terms of a series for $\sec x \tan x$. For what values of x should the series converge?

c. Check your result in part (b) by multiplying the series for $\sec x$ by the series given for $\tan x$ in Exercise 57.

59. Uniqueness of convergent power series

a. Show that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent and equal for all values of x in an open interval $(-c, c)$, then $a_n = b_n$ for every n. (*Hint:* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Differentiate term by term to show that a_n and b_n both equal $f^{(n)}(0)/(n!)$.)

b. Show that if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in an open interval $(-c, c)$, then $a_n = 0$ for every n.

60. The sum of the series $\sum_{n=0}^{\infty} (n^2/2^n)$ To find the sum of this series, express $1/(1-x)$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to $1/2$. What do you get?

61. The sum of the alternating harmonic series This exercise will show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

and

$$\lim_{n\to\infty} (h_{2n} - \ln 2n) = \gamma,$$

where γ is Euler's constant.

c. Use these facts to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{n\to\infty} s_{2n} = \ln 2.$$

62. Assume that the series $\sum a_n x^n$ converges for $x = 4$ and diverges for $x = 7$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

a. Converges absolutely for $x = -4$

b. Diverges for $x = 5$

c. Converges absolutely for $x = -8.5$

d. Converges for $x = -2$

e. Diverges for $x = 8$

f. Diverges for $x = -6$

g. Converges absolutely for $x = 0$

h. Converges absolutely for $x = -7.1$

63. Assume that the series $\sum a_n (x-2)^n$ converges for $x = -1$ and diverges for $x = 6$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

a. Converges absolutely for $x = 1$

b. Diverges for $x = -6$

c. Diverges for $x = 2$

d. Converges for $x = 0$

e. Converges absolutely for $x = 5$

f. Diverges for $x = 4.9$

g. Diverges for $x = 5.1$

h. Converges absolutely for $x = 4$

64. Proof of Theorem 21 Assume that $a = 0$ in Theorem 21 and that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $-R < x < R$. Let $g(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$. This exercise will prove that $f'(x) = g(x)$, that is, $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = g(x)$.

a. Use the Ratio Test to show that $g(x)$ converges for $-R < x < R$.

b. Use the Mean Value Theorem to show that

$$\frac{(x+h)^n - x^n}{h} = n c_n^{n-1}$$

for some c_n between x and $x + h$ for $n = 1, 2, 3, \ldots$.

c. Show that