1552

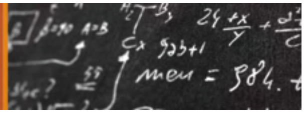
With Sal!

CHAPTER 5&8

* Calc 1 review
* Riemann Sums
* The Definite Integral
* Area between curves
* U-sub
* IBP
* trig integrals

* Calc 1 review
* Overview 1552
* $\boxed{\S 4.8}$

Math 1552
CALCULUS 1 REVIEW



Things you probably will need to know...

Special Cases for Limits at Infinity

If the degree of the numerator is greater than that of the denominator: $\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right) = \pm\infty$

If the degrees are equal: $\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right) = \frac{\text{leading coeff. of } P}{\text{leading coeff. of } Q}$

If the degree of the numerator is smaller than that of the denominator: $\lim_{x\to\infty}\left(\frac{P(x)}{Q(x)}\right) = 0$

... eventually.

Example:

Evaluate the following limits.

$$(a)\ \lim_{x\to\infty}\left(\frac{3+x^2}{9-5x^2}\right)$$
$$(b)\ \lim_{x\to\infty}\left(\frac{3x^2+5x^3}{7x^4-2x^3+4x^2}\right)$$

(a) $\lim_{x\to\infty} \frac{3+x^2}{9-5x^2} = \frac{-1}{5}$

(b) $\lim_{x\to\infty} \frac{3x^2+5x^3}{7x^4-2x^3+4x^2} = 0$

Derivative Rules

Power Rule: $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$

Ex. (a) $\frac{d}{dx} x^3 = 3x^2$

(b) $\frac{d}{dx} x^{-1} = -1 * x^{-2} = \frac{-1}{x^2}$

Example:

Differentiate the following functions:

(a) $f(x) = x^3\tan(x)$

(b) $g(x) = \frac{3\sec(x)}{2+x\cos(x)}$

(c) $h(x) = \arctan(\ln(5x))$

(a) $f'(x) = (x^3\tan(x))'$

$= (x^3)'\tan x + x^3(\tan x)'$ PROD

take $\frac{d}{dx}$

$= 3x^2\tan x + x^3\sec^2 x$

Remember this?

CHAIN rule

$[f(g(x))]' \overset{?}{=} f'(g(x)) * g'(x)$

(b) $g(x) = \left(\frac{3\sec x}{2+x\cos x}\right)'$

LO, D·HI, HI, D·LO

$= \frac{(2+x\cos x)(3\sec x)' - 3\sec x(2+x\cos x)'}{(2+x\cos x)^2}$

LO·LO

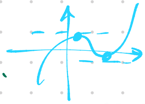
$= \frac{(2+x\cos x)(3\sec x\tan x) - 3\sec x(0 + 1\cos x - x(\sin x))}{(2+x\cos x)^2}$

(c) $h'(x) = \left(\arctan(\ln(5x))\right)'$

$= \frac{1}{1+(\ln(5x))^2} * (\ln(5x))'$ CHAIN

$= \frac{1}{1+(\ln(5x))^2} * \frac{1}{\cancel{5}x} * (\cancel{5x})'$ CHAIN x2

$= \frac{1}{x + x(\ln 5x)^2}$

It's all coming back...  ... or not.

Critical Number/Local Extrema

The number c is a *critical number* of f if

$$f'(c) = 0 \text{ or } f'(c) \text{ DNE}$$

We say the point (c, $f(c)$) is a *local extrema* if $f(c)$ is the smallest or largest value of f for all x-values "close" to c.

Increasing/Decreasing

A function f is said to be *increasing* on the interval (a,b) if $f'(x) > 0$ for all $x \in (a,b)$

A function f is said to be *decreasing* on the interval (a,b) if $f'(x) < 0$ for all $x \in (a,b)$

A function f is said to be *constant* on the interval (a,b) if $f'(x) = 0$ for all $x \in (a,b)$

Example:

Find all extreme values for the function below.

$$f(x) = (x-1)^2(x-2)^2 \text{ on } [0,4]$$

Soln. Need to find critical values where $f'(x)=0$ & then evaluate each region of $[0,4)$ between the critical values (e.g. use a sign chart).

$$f(x) = (x-1)^2(x-2)^2$$

$\frac{d}{dx}$ $\quad$ d/dx

$$f'(x) = 2(x-1)(x-2)^2 + (x-1)^2 \cdot 2(x-2)$$

$$= 2(x-1)(x-2)\left[(x-2) + (x-1)\right]$$

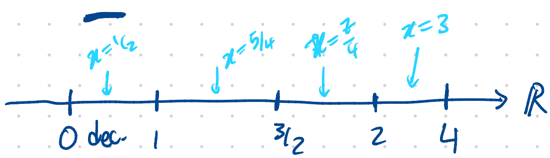
go back

$$= 2(x-1)(x-2)(2x-3) = 0$$

$$f'(x) = 0 \implies x = 1, 2, 3/2$$

$$f'(x) = 2(x-1)(x-2)(2x-3)$$

$x = 1/2$ $\quad$ $\frac{1}{2}-1$ $\quad$ $\frac{1}{2}-2$ $\quad$ $2\cdot\frac{1}{2}-3$



Ok, remembering now...

Absolute Extreme Values

Let f be continuous on $[a,b]$ and differentiable on (a,b). To find the ***absolute maximum*** and ***absolute minimum*** on $[a,b]$:

1. Find all critical numbers of f on $[a,b]$.
2. Evaluate $f(a)$, $f(b)$, and $f(c)$ for all critical numbers c.
3. The largest value in step 2 is the absolute maximum; the smallest value is the absolute minimum.

Do we need to know this...?

Some Derivative Formulas

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

Oh man, that's a lot of formulas,...

Derivatives of Inverse Functions

$$\frac{d}{dx}\left[\ln|u|\right] = \frac{1}{u}\cdot\frac{du}{dx} \qquad \frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[a^u\right] = a^u \ln(a)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\log_a u\right] = \frac{1}{u\ln(a)}\frac{du}{dx} \qquad \frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$$

Make it stop

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

OR

$$\frac{d}{dx}[\text{first} \times \text{second}]$$

$$= \left(\left[\frac{d}{dx}(\text{first})\right] \times \text{second}\right) + \left(\text{first} \times \left[\frac{d}{dx}(\text{second})\right]\right)$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

OR

$$\frac{d}{dx}\left[\frac{hi}{lo}\right] = \frac{lo \cdot d(hi) - hi \cdot d(lo)}{lo \cdot lo}$$

Is this on the exam?

Limit Theorems

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Then :

$(i)\ \lim_{x\to a}[f(x) \pm g(x)] = L \pm M$

$(ii)\ \lim_{x\to a} \alpha f(x) = \alpha L$, where $\alpha \in \Re$

$(iii)\ \lim_{x\to a}[f(x)g(x)] = LM$

$(iv)\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}, M \neq 0$

"optional"

Welcome to Math 1552

Syllabus Basics

- Your grade will be determined by:
  - Classwork Points (CP)
    - ~~Pre-lecture Videos~~
    - Online homework on MML
    - Studio Quizzes
  - Exams ~~(Friday evenings)~~ Thursday in Studio
    - Three 75-minute tests
    - For in-person sections exams are during studio (*QUP set up proctoring locally*)
    - Tests will be in person on June 8, June 29, July 20
  - Final Examination (not optional)
    - Thursday, July 28, from 11:20 am – 2:10 pm
    - Held in-person
    - Bonus points from CIOS and extra CP

Classwork Points *In-person ONLY*

Assignment	Maximum Number of Points
Pre-lecture videos	0 CP (36 videos – optional – not for a grade)
Start-of-semester survey and Syllabus Quiz	20 CP (10 points each)
Online problems on MML	37 CP (13 assignments, divide by 4 to convert MML points to CP)
Studio Quizzes	120 CP (6 quizzes, 20 points each)
Maximum total points for 100%:	130 out of 197
Extra credit	Every extra 1 CP over 130 is converted to 0.05 bonus point on the final exam (CIOS bonus an additional 5pt for a max total bonus of 8.35 bonus points on the final exam)

Grading Rubric *In-person ONLY*

Assessment	Weight
Classwork	20%
Midterm exams	55%
Final Exam*	25%

Classwork Points *QUP online ONLY*

Assignment	Maximum Number of Points
Pre-lecture videos	0 CP (36 videos – optional – not for a grade)
Start-of-semester survey, Syllabus Quiz and QUP Gradescope Quiz	30 CP (10 points each)
Online problems on MML	49 CP (13 assignments, divide by 3 to convert MML points to CP)
Studio Quizzes	120 CP (6 quizzes, 20 points each)
Maximum total points for 100%:	130 out of 199
Extra credit	Every extra 1 CP over 130 is converted to 0.05 bonus point on the final exam (CIOS bonus an additional 5pt for a max total bonus of 8.45 bonus points on the final exam)

Grading Rubric *QUP online ONLY*

Assessment	Weight
Classwork	05%
Midterm exams	70%
Final Exam*	25%

QUP section please do the Module 0 quiz in Canvas called "Proctoring Method Survey" by this Friday, May 19

Important Websites

- **Course Information:** canvas.gatech.edu
- **Textbook/Homework Access:**
  Use the "Pearson Access" tool on Canvas
- **On-line Discussions:** www.piazza.com
  (*highly recommended*)
- **Gradescope:**
  Use the "P~~earson Access" t~~ool on Canvas
  Gradescope.

Textbook: What to purchase?

- MyMathLab code is required to complete the online assignments.
  - You may sign up for temporary access for two weeks.
- IMPORTANT: Please register through **CANVAS**, not the MyMathLab site.

Important Policies

- Make-ups
  - NO MAKEUPS on studio quizzes, as there are extra points built into the classwork category
  - Contact me right away if you will miss an exam
  - One make-up policy
- Attendance
  - 1 CP per day of attendance in studio (in-person only) with active participation
- Calculators/websites/phones
  - Not allowed on any quizzes or exams
  - Zero on the assignment for first offense
  - **QUP online-only** students may use calculators for quizzes but not exams

Policies (cont)

- Academic Misconduct
  - Any cases will be submitted to the Dean's office.
- Disability Services
  - Please discuss any accommodations with me.
- Regrades
  - Submit on Gradescope within one week of receiving your graded paper.
  - Indicate which rubric item was not applied correctly

Math 1552

Section 4.8: Antiderivatives

anti-derivative.

$$x^2 \underset{d/dx}{\overset{\text{anti-derivative}}{\rightleftarrows}} 2x$$

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

Tentative Course Schedule

Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral

Antiderivatives

Definition: We say the function F is an *antiderivative* of the function f if $F'(x) = f(x)$.

Some useful formulas.

Function	*Antiderivative*
$ax^n, n \neq -1$	$a \cdot \frac{x^{n+1}}{n+1}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\csc^2(x)$	$-\cot(x)$
$\csc(x)\cot(x)$	$-\csc(x)$

anti-deriv.

d/dx

Function	*Antiderivative*
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$
$\sec(ax)\tan(ax)$	$\frac{1}{a}\sec(ax)$
$\csc^2(ax)$	$-\frac{1}{a}\cot(ax)$
$\csc(ax)\cot(ax)$	$-\frac{1}{a}\csc(ax)$

Find an anti-derivative for each function below.

(a) $f(x) = \sin(x) + \sqrt{x}$

(b) $g(x) = \frac{1}{4x^3} - \sec^2(x)$

(c) $h(x) = \left(x^3 - \frac{1}{x}\right)^2$

Example

(a) $f(x) = \sin(x) + \sqrt{x}$

$\frac{1}{1/2+1} x^{1/2+1} = \frac{1}{3/2} x^{3/2} = 2/3\, x^{3/2}$

why this?

So $F(x) = \boxed{-\cos x} + \boxed{\frac{2}{3}x^{3/2}} + C$

its derivative is $\sin x$

its derivative is $\sqrt{x}$

(a') $f(x) = \sin(3x)$

$F(x) = -\frac{1}{3}\cos(3x) + C$

(b) $g(x) = \frac{1}{4x^3} - \sec^2 x$

$\frac{1}{-3+1}\frac{1}{4} x^{-3+1} = \frac{-1}{8}x^{-2}$

$G(x) = \frac{-1}{8x^2} - \tan x + C$

(c) $h(x) = \left(x^3 - \frac{1}{x}\right)^2$ ← problem!

$h(x) = \left(x^3 - \frac{1}{x}\right)\left(x^3 - \frac{1}{x}\right) = x^3 \cdot x^3 - \underbrace{x^3 \cdot \frac{1}{x}}_{-x^2} - \underbrace{\frac{1}{x} x^3}_{-x^2} - \frac{1}{x} \cdot \frac{1}{x}$

$= x^6 - 2x^2 + \frac{1}{x^2} = x^6 - 2x^2 + x^{-2}$

$H(x) = \frac{1}{7}x^7 - \frac{2}{3}x^3 - x^{-1} + C$

$= \frac{1}{7}x^7 - \frac{2}{3}x^3 - \frac{1}{x} + C$

Check answer.

$H'(x) = \left(\frac{1}{7}x^7 - \frac{2}{3}x^3 - \frac{1}{x} + C\right)'$

$= \frac{1}{7} \cdot 7x^6 - \frac{2}{3} \cdot 3x^2 - \left(\frac{-1}{x^2}\right) + 0$

$= x^6 - 2x^2 + \frac{1}{x^2}$

Example 2:

A particle travels with an acceleration, in meters per second squared, given by:

$$a(t) = t - 5t^2.$$

Find the particle's initial velocity and initial position at time $t=0$ ~~t=2~~ second if the ~~initial position~~ is 2 m and the initial velocity is 10 m/s.

position at $t=1$ is 2
velocity at $t=1$ is 10

$v(t) =$ "anti-derivative of acceleration"

$$= \boxed{\tfrac{1}{2}t^2 - \tfrac{5}{3}t^3 + \cancel{C}\,\tfrac{67}{6}}$$

use $v(1) = 10$ m/s to find C.

$$v(1) = \tfrac{1}{2}(1)^2 - \tfrac{5}{3}(1)^3 + C = 10$$

$$\Rightarrow C = 10 - \tfrac{1}{2} + \tfrac{5}{3} = \frac{60 - 3 + 10}{6} = 67/6$$

Now Find the anti-derivative of $v(t)$

$$s(t) = \tfrac{1}{2}\cdot\tfrac{1}{3}t^3 - \tfrac{5}{3}\cdot\tfrac{1}{4}t^4 + \tfrac{67}{6}t + C$$

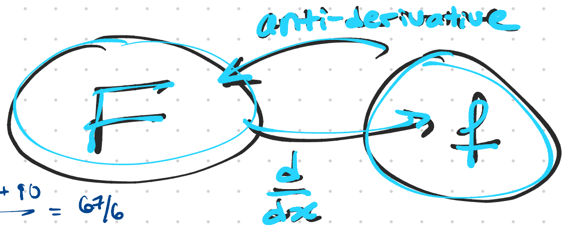
$$s(1) = \tfrac{1}{6}(1)^3 - \tfrac{5}{12}(1)^4 + \tfrac{67}{6}(1) + C = 2$$

$$\Longrightarrow C = 2 - \tfrac{1}{6} + \tfrac{5}{12} - \tfrac{67}{6} = \frac{24 - 2 + 5 - 134}{12}$$

$$= 26/3$$

$\frac{d}{dx}$ position = velocity

So the anti-derivative of velocity is position

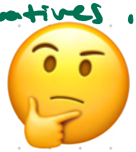


Ex $3x^2 \xrightarrow[d/dx]{} 6x$ (anti-derivative)

$$\begin{array}{r} 134 \\ -27 \\ \hline 104 \end{array} \qquad \frac{104}{12} = \frac{52}{6} = \frac{26}{3}$$

So this is why we needed to learn derivatives . . .

Example 3:



Evaluate each indefinite integral.

$$(a)\int\left(e^{-5x}+\sec x(\tan x-\sec x)\right)dx$$

$$(b)\int\left(\frac{1}{\sqrt{16-x^2}}-\frac{2}{x}\right)dx$$

$\frac{d}{dx}\sin^{-1}(ax)=$

Derivatives of Inverse Functions

$$\frac{d}{dx}[\ln|u|]=\frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[a^u]=a^u\ln(a)\frac{du}{dx}$$

$$\frac{d}{dx}[\log_a u]=\frac{1}{u\ln(a)}\frac{du}{dx}$$

*remember me...

$$\frac{d}{dx}[\sin^{-1}(ax)]=\frac{1}{\sqrt{1-(ax)^2}}\cdot a$$

$$\frac{d}{dx}[\cos^{-1}(x)]=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1}(x)]=\frac{1}{1+x^2}$$

Function	Antiderivative
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$
$\sec(ax)\tan(ax)$	$\frac{1}{a}\sec(ax)$
$\csc^2(ax)$	$-\frac{1}{a}\cot(ax)$
$\csc(ax)\cot(ax)$	$-\frac{1}{a}\csc(ax)$

$$\int\frac{1}{\sqrt{1-(ax)^2}}dx=\frac{1}{a}\sin^{-1}(ax)+C$$

ax

$\sec x\tan x-\sec^2 x$

$$(a)\int e^{-5x}+\sec x\cdot(\tan x-\sec x)\,dx$$

$$=-\frac{1}{5}e^{-5x}+\sec x-\tan x+C$$

Office Hours Poll

$$\int\frac{1}{x}dx=\ln|u|+C$$

$$\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}(x)+C$$

$$(b)\int\frac{1}{\sqrt{16-x^2}}-\frac{2}{x}\,dx$$

$$=\int\frac{1}{\sqrt{16(1-\frac{x^2}{16})}}-2\cdot\frac{1}{x}\,dx$$

$$=\int\frac{1}{4}\cdot\frac{1}{\sqrt{1-(\frac{x}{4})^2}}-2\frac{1}{x}\,dx$$

$$=\frac{1}{\cancel{4}}\cdot\frac{1}{\cancel{1/4}}\sin^{-1}(x/4)-2\ln|x|+C$$

Wait... what?

1. $\int_1^e x\ln(x^4)\,dx$

IBP

$\int x\,dx = \frac{1}{2}x^2 + C$

$\int x^4\,dx = \frac{1}{5}x^5 + C$

$\int \ln x\,dx = ?$

Function	Antiderivative
$\sin(ax)$	$-\frac{1}{a}\cos(ax)$
$\cos(ax)$	$\frac{1}{a}\sin(ax)$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)$
$\sec(ax)\tan(ax)$	$\frac{1}{a}\sec(ax)$
$\csc^2(ax)$	$-\frac{1}{a}\cot(ax)$
$\csc(ax)\cot(ax)$	$-\frac{1}{a}\csc(ax)$

Ummm...

2. $\int \sin^5(x)\cos^2(x)\,dx$

trig integral.

pythagorean + u-sub.

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

Derivatives of Inverse Functions

$$\frac{d}{dx}[\ln|u|] = \frac{1}{u}\cdot\frac{du}{dx} \qquad \frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[a^u] = a^u\ln(a)\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}}$$

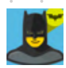
$$\frac{d}{dx}[\log_a u] = \frac{1}{u\ln(a)}\frac{du}{dx} \qquad \frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$$

you've got to be joking

3. $\int \frac{2}{x^2\sqrt{x^2-1}}\,dx$

not as hard as it looks

3. $\int \left(\frac{e^{\sqrt{2}}+x^{\sqrt{2}}}{\sqrt{x}}\right)dx$

Studio WS on Thursday 

$\int \frac{1}{1+(ax)^2}\,dx = \frac{1}{a}\tan^{-1}(ax) + C$

4. $\int \left(\frac{1}{1+9x^2}\right)dx$

$= \cancel{\frac{1}{9}\tan^{-1}(9x) + C}$

$(a=3)$

$= \int \frac{1}{1+(3x)^2}\,dx = \frac{1}{3}\tan^{-1}(3x) + C$

(c) $h(x) = \frac{1}{x} + 2e^{2x}$

Find

(1) The anti-derivative

(2) The average value over the interval [1,2]

(3) The definite integral of h(x) over the interval [1,2]

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	8. e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	9. $\frac{1}{x}$	$\ln \lvert x \rvert + C, \quad x \neq 0$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$	10. $\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	11. $\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	12. $\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	13. a^{kx}	$\left(\frac{1}{k \ln a}\right)a^{kx} + C, a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		

EXERCISES 4.8

Finding Antiderivatives

In Exercises 1–24, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. **a.** $2x$ **b.** x^2 **c.** $x^2 - 2x + 1$
2. **a.** $6x$ **b.** x^7 **c.** $x^7 - 6x + 8$
3. **a.** $-3x^{-4}$ **b.** x^{-4} **c.** $x^{-4} + 2x + 3$
4. **a.** $2x^{-3}$ **b.** $\frac{x^{-3}}{2} + x^2$ **c.** $-x^{-3} + x - 1$
5. **a.** $\frac{1}{x^2}$ **b.** $\frac{5}{x^2}$ **c.** $2 - \frac{5}{x^2}$
6. **a.** $-\frac{2}{x^3}$ **b.** $\frac{1}{2x^3}$ **c.** $x^3 - \frac{1}{x^3}$
7. **a.** $\frac{3}{2}\sqrt{x}$ **b.** $\frac{1}{2\sqrt{x}}$ **c.** $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. **a.** $\frac{4}{3}\sqrt[3]{x}$ **b.** $\frac{1}{3\sqrt[3]{x}}$ **c.** $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
9. **a.** $\frac{2}{3}x^{-1/3}$ **b.** $\frac{1}{3}x^{-2/3}$ **c.** $-\frac{1}{3}x^{-4/3}$
10. **a.** $\frac{1}{2}x^{-1/2}$ **b.** $-\frac{1}{2}x^{-3/2}$ **c.** $-\frac{3}{2}x^{-5/2}$
11. **a.** $\frac{1}{x}$ **b.** $\frac{7}{x}$ **c.** $1 - \frac{5}{x}$
12. **a.** $\frac{1}{3x}$ **b.** $\frac{2}{5x}$ **c.** $1 + \frac{4}{3x} - \frac{1}{x^2}$
13. **a.** $-\pi \sin \pi x$ **b.** $3 \sin x$ **c.** $\sin \pi x - 3 \sin 3x$
14. **a.** $\pi \cos \pi x$ **b.** $\frac{\pi}{2}\cos \frac{\pi x}{2}$ **c.** $\cos \frac{\pi x}{2} + \pi \cos x$
15. **a.** $\sec^2 x$ **b.** $\frac{2}{3}\sec^2 \frac{x}{3}$ **c.** $-\sec^2 \frac{3x}{2}$
16. **a.** $\csc^2 x$ **b.** $-\frac{3}{2}\csc^2 \frac{3x}{2}$ **c.** $1 - 8\csc^2 2x$
17. **a.** $\csc x \cot x$ **b.** $-\csc 5x \cot 5x$ **c.** $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
18. **a.** $\sec x \tan x$ **b.** $4 \sec 3x \tan 3x$ **c.** $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$
19. **a.** e^{3x} **b.** e^{-x} **c.** $e^{x/2}$
20. **a.** e^{-2x} **b.** $e^{4x/3}$ **c.** $e^{-x/5}$
21. **a.** 3^x **b.** 2^{-x} **c.** $\left(\frac{5}{3}\right)^x$
22. **a.** $x^{\sqrt{3}}$ **b.** x^{π} **c.** $x^{\sqrt{2}-1}$
23. **a.** $\frac{2}{\sqrt{1-x^2}}$ **b.** $\frac{1}{2(x^2+1)}$ **c.** $\frac{1}{1+4x^2}$
24. **a.** $x - \left(\frac{1}{2}\right)^x$ **b.** $x^2 + 2^x$ **c.** $\pi^x - x^{-1}$

Finding Indefinite Integrals

In Exercises 25–70, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

25. $\int (x + 1)\, dx$
26. $\int (5 - 6x)\, dx$
27. $\int \left(3t^2 + \frac{t}{2}\right) dt$
28. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$
29. $\int (2x^3 - 5x + 7)\, dx$
30. $\int (1 - x^2 - 3x^5)\, dx$
31. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$
32. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
33. $\int x^{-1/3}\, dx$
34. $\int x^{-5/4}\, dx$
35. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$
36. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
37. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$
38. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$
39. $\int 2x(1 - x^{-3})\, dx$
40. $\int x^{-3}(x + 1)\, dx$
41. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$
42. $\int \frac{4 + \sqrt{t}}{t^3}\, dt$
43. $\int (-2 \cos t)\, dt$
44. $\int (-5 \sin t)\, dt$
45. $\int 7 \sin \frac{\theta}{3}\, d\theta$
46. $\int 3 \cos 5\theta\, d\theta$
47. $\int (-3 \csc^2 x)\, dx$
48. $\int \left(-\frac{\sec^2 x}{3}\right) dx$
49. $\int \frac{\csc \theta \cot \theta}{2}\, d\theta$
50. $\int \frac{2}{5} \sec \theta \tan \theta\, d\theta$
51. $\int (e^{3x} + 5e^{-x})\, dx$
52. $\int (2e^x - 3e^{-2x})\, dx$
53. $\int (e^{-x} + 4^x)\, dx$
54. $\int (1.3)^x\, dx$
55. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$
56. $\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$
57. $\int (\sin 2x - \csc^2 x)\, dx$
58. $\int (2 \cos 2x - 3 \sin 3x)\, dx$
59. $\int \frac{1 + \cos 4t}{2}\, dt$
60. $\int \frac{1 - \cos 6t}{2}\, dt$

Math 1552

Sections 5.1-5.3:
Area under the Curve
The Definite Integral

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

Tentative Course Schedule

Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral

you are Here.

Day 1 Learning Goals

- Understand how to partition an interval
- Draw a picture to approximate the area under the curve with a given number of rectangles
- Compute the Upper and Lower sums
- Calculate the midpoint estimate

Riemann Sums

- Idea: Find the area bounded by a function $f(x)$, the lines $x=a$, $x=b$, and the x-axis.
- Procedure: Break the interval $[a,b]$ into n subintervals, and draw a rectangle in each subinterval.
- Summing the areas of the rectangles will approximate the area under the curve.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

Example 1:

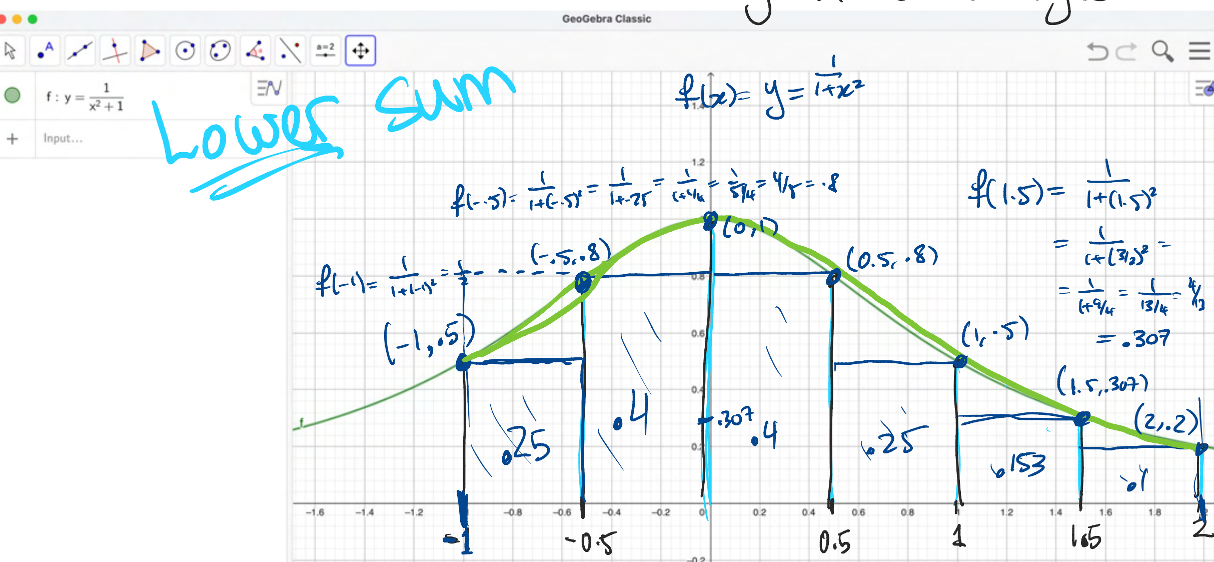
Find the upper and lower sums for the function

$$f(x) = \frac{1}{x^2+1}$$

on the interval [-1,2] with n=6 subintervals.

Ex $f(x) = \frac{1}{1+x^2}$ over $[-1,2]$

using $n=6$ rectangles.



Area $\approx (.5)(.5) + (.5)(.8) + (.5)(.8) + (.5)(.5) + (.5)(.307) + (.5)(.2)$

$= \boxed{1.553}$ (under-estimate)

Over-estimate / under-estimate / left / right

↑ always pick larger height (y-value)

always pick the lower height (y-value)

left always pick left y-value

right always pick right y-value.

Midpoint Estimate

Plug in the midpoint of each subinterval.

On the subinterval $[x_{i-1}, x_i]$,

the midpoint is: $\frac{x_{i-1}+x_i}{2}$

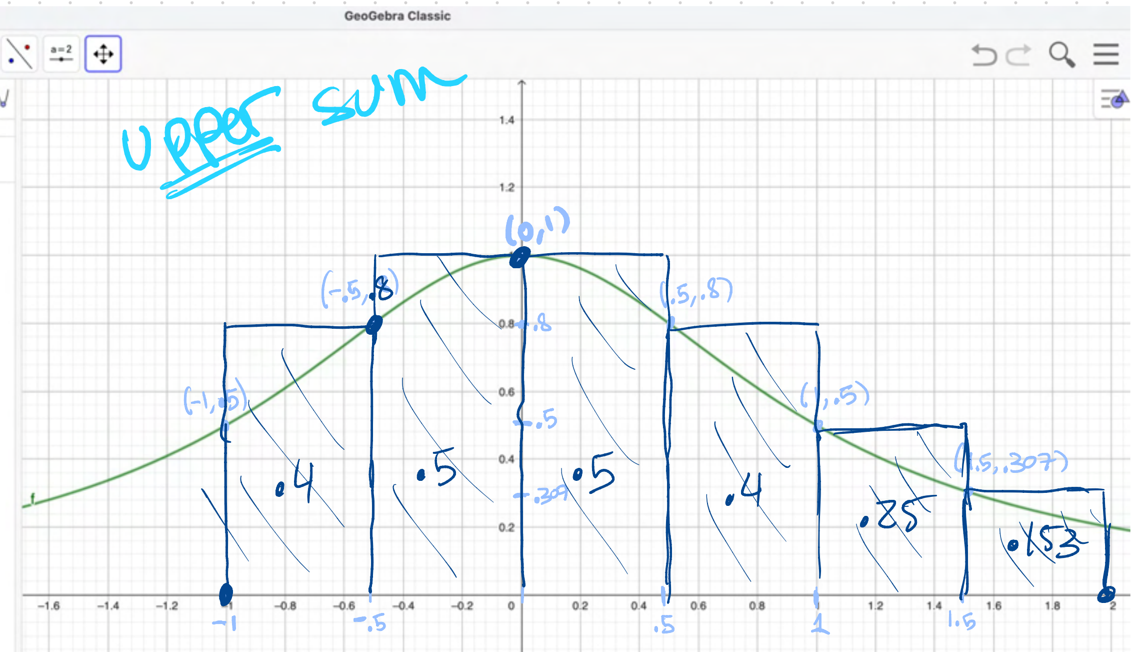
and the midpoint sum is:

$$M_f = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

Average Value

The ***average value*** of f on $[a,b]$ is the y-value that would generate a rectangle with the same area as f on $[a,b]$.

$$AV = \frac{Area}{b-a}$$



$$(.5)(.8) + (.5)(.1) + (.5)(.1) + (.5)(.8) + (.5)(.307)$$

$$= .4 + .5 + .5 + .4 + .25 + .153$$

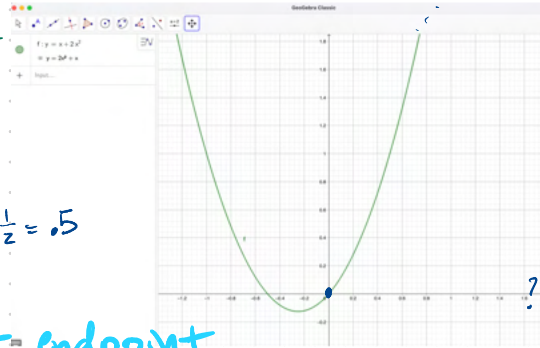
$$= .8 + .1 + .403 = 2.203$$ over-estimate

(1.553 under-estimate)

Ex **2.** Consider the function $f(x) = x + 2x^2$ on the interval $[0, 2]$. Using ~~a midpoint estimate~~ with $n = 4$ subintervals, ~~estimate the average value of f.~~

(a) left endpoint

(b) right endpoint



Notation

n = # intervals $= 4$

a = interval left endpoint $= 0$

b = interval right endpoint $= 2$

→ x_0 = first left endpoint

→ x_n = last right endpoint

Δx = width of each rectangle $= \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} = .5$

META

① Identify $[a,b]$ & n in the problem.

& compute the x_i values.

② plug in x_i-values into $f(x)$ to get y-values.

③ mult. y-values by width & add up.

Left endpoint

$f(x) = y = x + 2x^2$

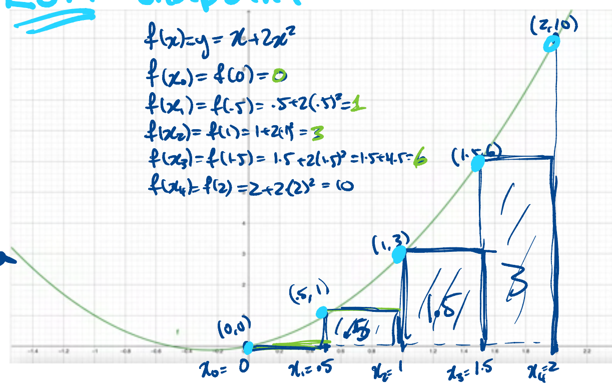
$f(x_0) = f(0) = 0$

$f(x_1) = f(.5) = .5 + 2(.5)^2 = 1$

$f(x_2) = f(1) = 1 + 2(1)^2 = 3$

$f(x_3) = f(1.5) = 1.5 + 2(1.5)^2 = 1.5 + 4.5 = 6$

$f(x_4) = f(2) = 2 + 2(2)^2 = 10$



$(0.5)(0) + .5(1) + .5(3) + .5(6)$

$= (0.5)[0 + 1 + 3 + 6]$

$= (.5)(10) = \boxed{5}$

Ex.

Approx area under $f(x) = x + 2x^2$ over $[0,2]$ w/ $n = 4$ rectangles using right-hand endpoint rule.

Right endpoint

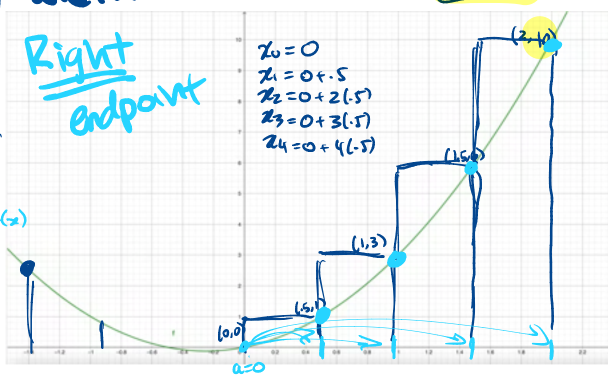
$x_0 = 0$

$x_1 = 0 + .5$

$x_2 = 0 + 2(.5)$

$x_3 = 0 + 3(.5)$

$x_4 = 0 + 4(.5)$



Soln ① Find endpoints. ② plug in x-values into $f(x)$

$x_1 = .5$ $\quad f(.5) = .5 + 2(.5)^2 = 1$

$x_2 = 1$ $\quad f(1) = 3$

$x_3 = 1.5$ $\quad f(1.5) = 6$

$x_4 = 2$ $\quad f(2) = 10$

③ Area $\approx .5(1) + .5(3) + .5(6) + .5(10)$

$= (.5)[1 + 3 + 6 + 10]$

$= .5(20) = 10$

$.5(1) + .5(3) + .5(6) + .5(8)$

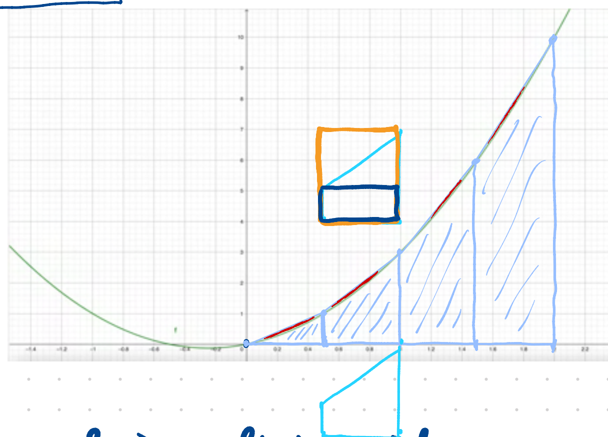
$= .5(1 + 3 + 6 + 8)$

$= .5(18) = 9$

Ex. **2.** Consider the function $f(x) = x + 2x^2$ on the interval $[0, 2]$. Using a ~~midpoint estimate~~ with $n = 4$ subintervals, ~~estimate the average value of f.~~

(a) left endpoint

$\boxed{\text{(b) right endpoint}}$



Ex. 3

$$L_4 = \sum_{k=0}^{3} f(x_k)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$R_4 = \sum_{k=1}^{4} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

Notation notes!

OOPS! TRAPEZOID RULE

$$\frac{L_4 + R_4}{2} = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

left endpoint formula

$$L_n = \sum_{k=0}^{n-1} f(x_k)\Delta x$$

Δx = width of each rectangle $= \dfrac{b-a}{n}$

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\cdot\Delta x$

$\vdots$

$x_k = a + k\cdot\Delta x$

$\vdots$

$x_n = a + n\cdot\Delta x = a + \cancel{n}\left(\dfrac{b-a}{\cancel{n}}\right)$

$= a + b - a = b$

right endpoint formula

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

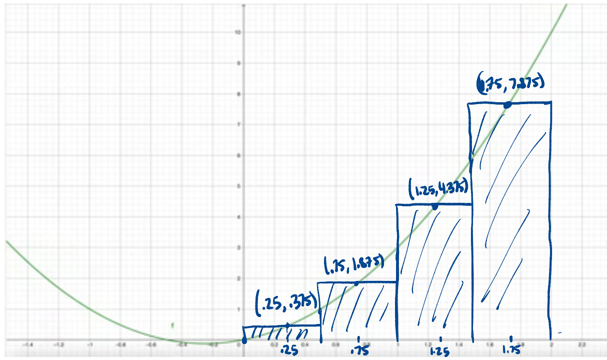
midpoint formula

Ex. 2. Consider the function $f(x) = x + 2x^2$ on the interval $[0,2]$. Using a midpoint estimate with $n = 4$ subintervals, estimate the *average value* of f.

Friday Start HERE

BIG OOPS trapezoid

not midpoint



$(.5)(.375)+(.5)(1.875)+(.5)(4.375)+.5(7.875)$

$= .5(.375+1.875+4.375+7.875)$

$= .5(14.5) = 7.25$

Notation notes:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

$$\Delta x = \frac{b-a}{n}$$

$$x_0 = a$$

$$x_1 = a + \Delta x$$

$\vdots$

$$x_k = a + k \cdot \Delta x$$

$\vdots$

$$x_n = a + n \cdot \Delta x = \boxed{b}$$

$$= a + \cancel{n}\left(\frac{b-a}{\cancel{n}}\right)$$

$$= a + b - a = \boxed{b}$$

left endpoint formula

$$L_n = \sum_{k=0}^{n-1} \Delta x \cdot f(x_k)$$

$$= \Delta x\, f(x_0) + \Delta x\, f(x_1) + \ldots + \Delta x\, f(x_{n-1})$$

$$= \Delta x\left(f(x_0) + f(x_1) + \ldots + f(x_{n-1})\right)$$

right endpoint formula

$$R_n = \sum_{k=1}^{n} \Delta x\, f(x_k)$$

$$= \Delta x\, f(x_1) + \Delta x\, f(x_2) + \ldots + \Delta x\, f(x_n)$$

$$= \Delta x\left(f(x_1) + f(x_2) + \ldots + f(x_n)\right)$$

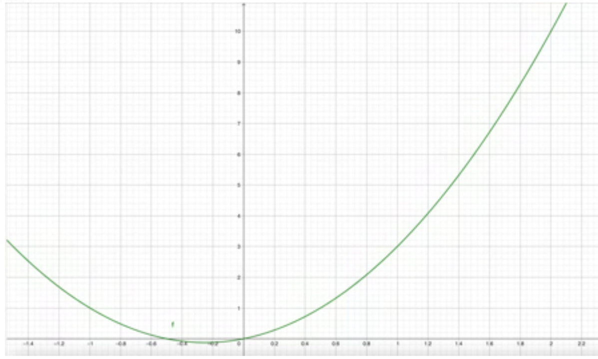
midpoint formula

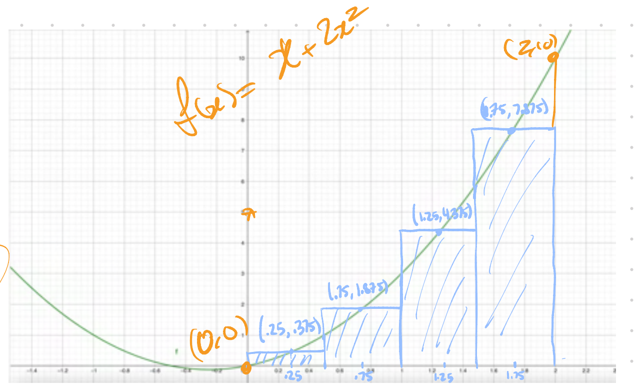
$$M_n = \frac{L_n + R_n}{2} = \sum_{k=0}^{n-1} \Delta x\left[\frac{f(x_k) + f(x_{k+1})}{2}\right]$$

$$= \sum_{k=0}^{n-1} \frac{\Delta x}{2}\left(f(x_k) + f(x_{k+1})\right)$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

Ex. **2.** Consider the function $f(x) = x + 2x^2$ on the interval $[0,2]$. Using a midpoint estimate with $n = 4$ subintervals, estimate the *average value* of f.





approx avg value

$$7.25 / 2$$

$$= 3.125$$

$$(.5)(.375) + (.5)(1.875) + (.5)(4.375) + .5(7.875)$$

$$= .5(.375 + 1.875 + 4.375 + 7.875)$$

$$= .5(14.5) = 7.25$$

So: (a) What was the lower estimate?
(From left endpoints)

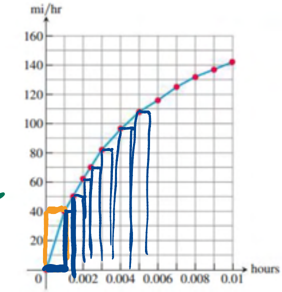
(b) what was upper estimate?
(From right endpoints)

(c) and the midpoint estimate gave us what value?

Note to self
Don't forget average value!

Ex 3 **Distance from velocity data** The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		



total distance

a. Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b. ~~Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?~~

b. estimate the average speed over the 36 second trip.

Q1: What is Δx?

Q2: What is a? what is b?

Q3: What are the $f(x_i)$ values?

$$.001\left(0+40+62+82+96+108+116+125+132+137\right)$$

$$\frac{\text{total dist}}{\text{total time}} = \text{avg speed}$$

$n=10$

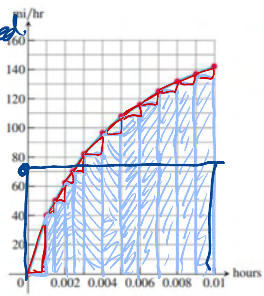
$\Delta x = .001 \quad x_0 = 0, \; x_{10} = 0.010$

$$\text{Area} \approx \Delta x\left(f(x_0)+f(x_1)+f(x_2)+\dots+f(x_{n-1})\right)$$

$$= (.001)\left[0+40+62+82+96+108+116+125+132+137+\cancel{142}\right]$$

(not this one.

$= (.001)(898) = .898$ miles traveled



$$\frac{.898}{.01}\frac{\cancel{100}}{100} = \frac{89.8}{1}$$

a. Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b. Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

$$\text{avg value of } f(x) \text{ over } [a,b] = \frac{\text{area under } f(x) \text{ over } [a,b]}{\text{length of } [a,b]}$$

EXERCISES 5.1

Area

In Exercises 1–4, use finite approximations to estimate the area under the graph of the function using

a. a lower sum with two rectangles of equal width.
b. a lower sum with four rectangles of equal width.
c. an upper sum with two rectangles of equal width.
d. an upper sum with four rectangles of equal width.

1. $f(x) = x^2$ between $x = 0$ and $x = 1$.
2. $f(x) = x^3$ between $x = 0$ and $x = 1$.
3. $f(x) = 1/x$ between $x = 1$ and $x = 5$.
4. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

5. $f(x) = x^2$ between $x = 0$ and $x = 1$.
6. $f(x) = x^3$ between $x = 0$ and $x = 1$.
7. $f(x) = 1/x$ between $x = 1$ and $x = 5$.
8. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Distance

9. **Distance traveled** The accompanying table shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

a. left-endpoint values.
b. right-endpoint values.

Time (sec)	Velocity (cm/sec)	Time (sec)	Velocity (cm/sec)
0	0	6	28
1	30	7	15
2	56	8	5
3	25	9	15
4	38	10	0
5	33		

10. **Distance traveled upstream** You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

MOVE ME

5.2 Sigma Notation and Limits of Finite Sums

unless, you know... summer

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

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you are here

Days 2 & 3 Learning Goals

- Be able to find the equation for a general Riemann Sum
- Take the limit of your answer to find the actual area beneath the curve
- Understand the definition of the definite integral
- Understand key properties of the definite integral

EXAMPLE 0

A sum in sigma notation

(a) $\sum_{k=1}^{5} k$

(b) $\sum_{k=1}^{3} (-1)^k k$

(c) $\sum_{k=1}^{2} \frac{k}{k+1}$

(d) $\sum_{k=4}^{5} \frac{k^2}{k-1}$

Warm-up

$$\sum_{k=1}^{5} k = 1+2+3+4+5 = 15$$

$$\sum_{k=1}^{3} (-1)^k k = \underbrace{(-1)^1 \cdot 1}_{k=1} + \underbrace{(-1)^2 2}_{k=2} + \underbrace{(-1)^3 \cdot 3}_{} = -1 + 2 - 3 = -2$$

$$\sum_{k=1}^{2} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} = \frac{1}{2} + \frac{2}{3} = \frac{3+4}{6} = 7/6$$

$$\sum_{k=4}^{5} \frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{16 \cdot 4 + 25 \cdot 3}{12} = \frac{64+75}{12} = \frac{139}{12}$$

First example of the "day"

Ex 1 A marketing company is trying a new campaign. The campaign lasts for three weeks, and during this time, the company finds that it gains customers as a function of time according to the formula:

$$\boxed{C(t) = 5t - t^2,} \quad [0,3]$$

where t is time in weeks and the number of customers is given in thousands. Using the general form of the definite integral,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*),$$

calculate the **average** number of customers gained per week during the three-week campaign.

General Riemann Sum

Partition the interval $[a,b]$ into n equal pieces:

$a = x_0 < x_1 < x_2 < .... < a_n = b$

Let x_i^* be an arbitrary point in the interval $[x_{i-1}, x_i]$.

Then we can estimate the area under the curve between $x = a$ and $x = b$ with the formula:

$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$

Note that: $L_f \le A \le U_f$

Steps

① First find the TOTAL number of customers gained ← approx using ① RS. then taking ② limit.

② divide by # of weeks to get average per week.

What is x_i*?

A. The left-hand endpoint of the subinterval.
B. The right-hand endpoint of the subinterval.
C. The midpoint of the subinterval.
D. Any value on the subinterval.

Meta for RS approximation

① Find x-values

② plug in x-values into function to get y-values

③ mult. y-values * Δx & add up.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = 3/n$$

$$x_0 = a = 0$$

$$x_1 = a + \Delta x = 0 + 3/n$$

$$x_2 = x_1 + \Delta x = a + 2 * 3/n = \frac{3 \cdot 2}{n}$$

$$\vdots$$

$$x_k = a + k \cdot \Delta x = 0 + 3k/n$$

$$\vdots$$

$$x_n = b = 3 = 0 + n \cdot \frac{3}{n}$$

First example of the "day"

Ex 1 A marketing company is trying a new campaign. The campaign lasts for three weeks, and during this time, the company finds that it gains customers as a function of time according to the formula:

$$\boxed{C(t) = 5t - t^2}, \quad x_k = \frac{3k}{n}$$

where t is time in weeks and the number of customers is given in thousands. Using the general form of the definite integral,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*),$$

calculate the **average** number of customers gained during the three-week campaign. per week

$x_k = \frac{3k}{n}$ $\quad \Delta x = \frac{3}{n}$

$$R_n = \sum_{k=1}^{n} f(x_k)\cdot \Delta x$$

!?

$$= \sum_{k=1}^{n} \left(5\cdot\left(\frac{3k}{n}\right) - \left(\frac{3k}{n}\right)^2\right) * \left(\frac{3}{n}\right)$$

$$= \sum_{k=1}^{n} \left(\frac{15k}{n} - \frac{9k^2}{n^2}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{k=1}^{n} \frac{45k}{n^2} - \frac{27k^2}{n^3}$$

Steps

1. First find the TOTAL number of customers gained
2. divide by # of weeks to get average per week.

$$\text{Total} \approx \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(5\left(\frac{3k}{n}\right) - \left(\frac{3k}{n}\right)^2\right)\left(\frac{3}{n}\right)$$

$$= \sum_{k=1}^{n}\left(\frac{15k}{n} - \frac{9k^2}{n^2}\right)\cdot\left(\frac{3}{n}\right)$$

$$R_n = \sum_{k=1}^{n} \frac{45k}{n^2} - \frac{27k^2}{n^3}$$

need to take the limit as $n \to \infty$

① $$= \sum_{k=1}^{n}\frac{45k}{n^2} - \sum_{k=1}^{n}\frac{27k^2}{n^3}$$

② $$= \frac{45}{n^2}\sum_{k=1}^{n} k - \frac{27}{n^3}\sum_{k=1}^{n} k^2$$

② $\frac{45}{n^2}(1+2+3+4+5)$ $\quad \frac{5(5+1)}{2}$

③ $$= \frac{45}{n^2}\cdot\frac{n(n+1)}{2} - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{45}{2}\cdot\frac{n(n+1)}{n^2} - \frac{27}{6}\cdot\frac{n(n+1)(2n+1)}{n^3}$$

$(n^2+n)(2n+1)$

$$\Rightarrow \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{45}{2}\underbrace{\frac{n(n+1)}{n^2}}_{1} - \frac{27}{6}\underbrace{\frac{n(n+1)(2n+1)}{n^3}}_{2}\right)$$

$n^2+n \qquad 2n^3+3n^2+n$

$$= \frac{45}{2}\cdot 1 - \frac{27}{6}\cdot 2 = \frac{45}{2} - \frac{27}{3} = \frac{45}{2} - 9$$

$$= 13.5$$

② $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

① $$\sum_{k=1}^{n} k \overset{?}{=} 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{100} k = 50\cdot 101 = 5050$$

$1+2+3+4+\dots+97+98+99+100$

③ $$\sum_{k=1}^{n} 1 = n$$

(b) avg # customers gained / per week

$13.5/3 = 4.5$

Example 2:

Use the method of Riemann Sums to evaluate the following definite integral. Choose x_i^* to be the right-hand endpoint of each subinterval.

$$\int_{-1}^{2} (x+1)^2\,dx$$

$f(x) = (x+1)^2$

$a=-1$ $b=2$

$[-1, 2]$ $\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

① Find the x-values.

$x_0 = -1 = a$

$x_1 = -1 + 3/n$

$x_2 = -1 + 3/n * 2$

$\vdots$

$x_k = -1 + \frac{3k}{n}$

$\vdots$

$x_n = -1 + \frac{3n}{n} = -1+3 = 2 = b$

$f(x) = (x+1)^2$

The Definite Integral

We define the definite integral to be the limit of the Riemann Sum: $\frac{1}{3} = .3333\ldots$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Helpful Summation Formulas

$\sum_{i=1}^{n} 1 = n$

① $\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$ ← how to deal w/ x

② $\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$ ← how to deal w/ x^2

$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$\sum_{i=1}^{n} c a_i = c\sum_{i=1}^{n} a_i$

② get y-values & ③ plug into RS formula.

$$R_n = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n}\Big(\overbrace{\cancel{-1} + \frac{3k}{n}}^{x_k} \cancel{+1}\Big)^2 * \overset{\Delta x}{\frac{3}{n}}$$

$$= \sum_{k=1}^{n}\left(\frac{3k}{n}\right)^2 \cdot \frac{3}{n} = \sum_{k=1}^{n} \frac{9k^2 \cdot 3}{n^2 \cdot n}$$

$$= \sum_{k=1}^{n} \frac{27k^2}{n^3}$$

$$= \frac{27}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} \xrightarrow[n\to\infty]{} \frac{27}{6}\cdot\frac{2}{3} = 9$$

$\int_{-1}^{2} (x+1)^2\,dx = \int_{-1}^{2} x^2+2x+1\,dx$

$= \frac{1}{3}x^3 + x^2 + x \Big|_{-1}^{2}$

$= (8/3 + 4 + 2) - (-1/3 + 1 - 1)$

$= \frac{8}{3} + 6 - (-1/3) = \cancel{29/3}$ 9

Start HERE on Monday if more practice is needed.

Example 3:

In a memory experiment, the rate of memorization is measured by the function:

$f(t) = -0.006\,t^2 + 0.2\,t$,

where t is the time in minutes, and $f(t)$ is the number of words per minute.

(a) How many words are memorized in the first 20 minutes (from t=0 to t=20)? USE RIEMANN SUMS.

(b) What is the *average* number of words memorized each minute?

$f(t) = -0.006t^2 + 0.2t$

So here is the theory

The Definite Integral

We define the definite integral to be the limit of the Riemann Sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

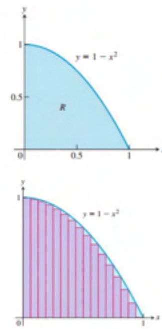
Check your understanding

True or False?

- The definite integral represents the total area bounded by the function, the lines x=*a* and x=*b*, and the x-axis.

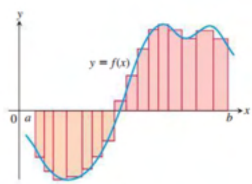
The Definite Integral and Area

If the function is always non-negative on [a,b], we have found *TOTAL AREA* under the curve.



(a)

If the function takes on negative values, then we have found the *NET AREA* under the curve.



(a)

EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\sum_{k=1}^{2} \frac{6k}{k+1}$
2. $\sum_{k=1}^{3} \frac{k-1}{k}$
3. $\sum_{k=1}^{4} \cos k\pi$
4. $\sum_{k=1}^{5} \sin k\pi$
5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$
6. $\sum_{k=1}^{4} (-1)^k \cos k\pi$
7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

   a. $\sum_{k=1}^{6} 2^{k-1}$ **b.** $\sum_{k=0}^{5} 2^k$ **c.** $\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

   a. $\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\sum_{k=0}^{5} (-1)^k 2^k$ **c.** $\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

   a. $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\sum_{k=0}^{2} \frac{(-1)^k}{k+1}$ **c.** $\sum_{k=-1}^{1} \frac{(-1)^k}{k+2}$

10. Which formula is not equivalent to the other two?

    a. $\sum_{k=1}^{4} (k-1)^2$ **b.** $\sum_{k=-1}^{3} (k+1)^2$ **c.** $\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice for the starting index.

11. $1 + 2 + 3 + 4 + 5 + 6$
12. $1 + 4 + 9 + 16$
13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
14. $2 + 4 + 6 + 8 + 10$
15. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$
16. $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

    a. $\sum_{k=1}^{n} 3a_k$ **b.** $\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\sum_{k=1}^{n} (a_k + b_k)$

    d. $\sum_{k=1}^{n} (a_k - b_k)$ **e.** $\sum_{k=1}^{n} (b_k - 2a_k)$

18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

    a. $\sum_{k=1}^{n} 8a_k$ **b.** $\sum_{k=1}^{n} 250b_k$

    c. $\sum_{k=1}^{n} (a_k + 1)$ **d.** $\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

19. **a.** $\sum_{k=1}^{10} k$ **b.** $\sum_{k=1}^{10} k^2$ **c.** $\sum_{k=1}^{10} k^3$
20. **a.** $\sum_{k=1}^{13} k$ **b.** $\sum_{k=1}^{13} k^2$ **c.** $\sum_{k=1}^{13} k^3$
21. $\sum_{k=1}^{7} (-2k)$
22. $\sum_{k=1}^{5} \frac{\pi k}{15}$
23. $\sum_{k=1}^{6} (3 - k^2)$
24. $\sum_{k=1}^{6} (k^2 - 5)$
25. $\sum_{k=1}^{5} k(3k + 5)$
26. $\sum_{k=1}^{7} k(2k + 1)$
27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$
28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$
29. **a.** $\sum_{k=1}^{7} 3$ **b.** $\sum_{k=1}^{500} 7$ **c.** $\sum_{k=3}^{264} 10$
30. **a.** $\sum_{k=9}^{36} k$ **b.** $\sum_{k=3}^{17} k^2$ **c.** $\sum_{k=18}^{71} k(k-1)$
31. **a.** $\sum_{k=1}^{n} 4$ **b.** $\sum_{k=1}^{n} c$ **c.** $\sum_{k=1}^{n} (k - 1)$
32. **a.** $\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ **b.** $\sum_{k=1}^{n} \frac{c}{n}$ **c.** $\sum_{k=1}^{n} \frac{k}{n^2}$
33. $\sum_{k=1}^{50} [(k+1)^2 - k^2]$
34. $\sum_{k=2}^{20} [\sin(k-1) - \sin k]$
35. $\sum_{k=7}^{30} (\sqrt{k-4} - \sqrt{k-3})$
36. $\sum_{k=1}^{40} \frac{1}{k(k+1)}$ $\left(\textit{Hint: } \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}\right)$

Riemann Sums

In Exercises 37–42, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

37. $f(x) = x^2 - 1,\ [0, 2]$
38. $f(x) = -x^2,\ [0, 1]$
39. $f(x) = \sin x,\ [-\pi, \pi]$
40. $f(x) = \sin x + 1,\ [-\pi, \pi]$
41. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.
42. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Riemann Sums

For the functions in Exercises 43–50, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

43. $f(x) = 1 - x^2$ over the interval $[0, 1]$.
44. $f(x) = 2x$ over the interval $[0, 3]$.
45. $f(x) = x^2 + 1$ over the interval $[0, 3]$.
46. $f(x) = 3x^2$ over the interval $[0, 1]$.
47. $f(x) = x + x^2$ over the interval $[0, 1]$.
48. $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.
49. $f(x) = 2x^3$ over the interval $[0, 1]$.
50. $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.