Start HERE on Monday.
If more practice is needed.

Example 3:

In a memory experiment, the rate of memorization is measured by the function:

$f(t) = -0.006\,t^2 + 0.2\,t$

where t is the time in minutes, and $f(t)$ is the number of words per minute.

(a) How many words are memorized in the first 20 minutes (from t=0 to t=20)? USE RIEMANN SUMS.

(b) What is the *average* number of words memorized each minute?

So here is the theory

The Definite Integral

We define the definite integral to be the limit of the Riemann Sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Check your understanding

True or False?

- The definite integral represents the total area bounded by the function, the lines x=a and x=b, and the x-axis.

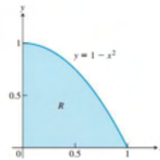
$f(t) = -0.006t^2 + 0.2t$ over $[0,20]$

META. ① Find Δx & x_k

② plug in x_k into $f(x)$

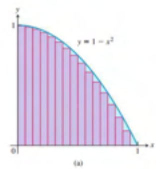
③ get rid of Σ's in $R_n = \sum_{k=1}^{n} f(x_k)\,\Delta x$

④ take limit.



The Definite Integral and Area

If the function is always non-negative on [a,b], we have found *TOTAL AREA* under the curve.



If the function takes on negative values, then we have found the *NET AREA* under the curve.

①
$$\Delta x = \frac{b-a}{n} = \frac{20-0}{n} = \frac{20}{n}$$
$$x_k = a + k\cdot\Delta x = 0 + k\cdot\frac{20}{n} = \frac{20k}{n}$$

② $$f(x_k) = f\left(\frac{20k}{n}\right) = -0.006\left(\frac{20k}{n}\right)^2 + 0.2\left(\frac{20k}{n}\right)$$

$$= (-0.006)\frac{400k^2}{n^2} + (.2)\frac{20k}{n}$$

$$f(x_k) = -\frac{2.4k^2}{n^2} + \frac{4k}{n}$$

① $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

② $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

③ $\sum_{k=1}^{n} 1 = n$

③ $$R_n = \sum_{k=1}^{n} \underbrace{\left(-\frac{2.4k^2}{n^2} + \frac{4k}{n}\right)}_{f(x_k)} \cdot \underbrace{\frac{20}{n}}_{\Delta x}$$

$$= \sum_{k=1}^{n} \frac{-2.4(20)k^2}{n^3} + \frac{4(20)k}{n^2}$$

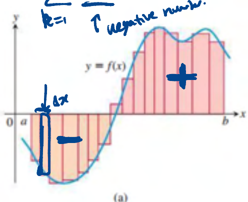
$$= -\frac{48}{n^3}\sum_{k=1}^{n} k^2 + \frac{80}{n^2}\sum_{k=1}^{n} k$$

$$= -\frac{48}{n^3}\,\frac{\overbrace{n(n+1)(2n+1)}^{2n^3+3n^2+n}}{6} + \frac{80}{n^2}\,\frac{\overbrace{n(n+1)}^{n^2+n}}{2} \xrightarrow{n\to\infty} -\frac{48}{6}\cdot 2 + \frac{80}{2}\cdot 1 = -16 + 40$$

(a) $= \boxed{24}$

(b) $\frac{24}{20}$ = avg words/min $= \boxed{1.2}$

$\sum_{k=1}^{n} f(x_k^*)\,\Delta x$
↑ negative number.



EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\sum_{k=1}^{2} \frac{6k}{k+1}$
2. $\sum_{k=1}^{3} \frac{k-1}{k}$
3. $\sum_{k=1}^{4} \cos k\pi$
4. $\sum_{k=1}^{5} \sin k\pi$
5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$
6. $\sum_{k=1}^{4} (-1)^{k} \cos k\pi$
7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

   a. $\sum_{k=1}^{6} 2^{k-1}$ **b.** $\sum_{k=0}^{5} 2^{k}$ **c.** $\sum_{k=-1}^{4} 2^{k+1}$
8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

   a. $\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\sum_{k=0}^{5} (-1)^{k} 2^{k}$ **c.** $\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$
9. Which formula is not equivalent to the other two?

   a. $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ **c.** $\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$
10. Which formula is not equivalent to the other two?

    a. $\sum_{k=1}^{4} (k-1)^2$ **b.** $\sum_{k=-1}^{3} (k+1)^2$ **c.** $\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice for the starting index.

11. $1 + 2 + 3 + 4 + 5 + 6$
12. $1 + 4 + 9 + 16$
13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
14. $2 + 4 + 6 + 8 + 10$
15. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$
16. $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

    a. $\sum_{k=1}^{n} 3a_k$ **b.** $\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\sum_{k=1}^{n} (a_k + b_k)$

    d. $\sum_{k=1}^{n} (a_k - b_k)$ **e.** $\sum_{k=1}^{n} (b_k - 2a_k)$
18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

    a. $\sum_{k=1}^{n} 8a_k$ **b.** $\sum_{k=1}^{n} 250b_k$

    c. $\sum_{k=1}^{n} (a_k + 1)$ **d.** $\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

19. **a.** $\sum_{k=1}^{10} k$ **b.** $\sum_{k=1}^{10} k^2$ **c.** $\sum_{k=1}^{10} k^3$
20. **a.** $\sum_{k=1}^{13} k$ **b.** $\sum_{k=1}^{13} k^2$ **c.** $\sum_{k=1}^{13} k^3$
21. $\sum_{k=1}^{7} (-2k)$
22. $\sum_{k=1}^{5} \frac{\pi k}{15}$
23. $\sum_{k=1}^{6} (3 - k^2)$
24. $\sum_{k=1}^{6} (k^2 - 5)$
25. $\sum_{k=1}^{5} k(3k + 5)$
26. $\sum_{k=1}^{7} k(2k + 1)$
27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$
28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$
29. **a.** $\sum_{k=1}^{7} 3$ **b.** $\sum_{k=1}^{500} 7$ **c.** $\sum_{k=3}^{264} 10$
30. **a.** $\sum_{k=9}^{36} k$ **b.** $\sum_{k=3}^{17} k^2$ **c.** $\sum_{k=18}^{71} k(k-1)$
31. **a.** $\sum_{k=1}^{n} 4$ **b.** $\sum_{k=1}^{n} c$ **c.** $\sum_{k=1}^{n} (k-1)$
32. **a.** $\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ **b.** $\sum_{k=1}^{n} \frac{c}{n}$ **c.** $\sum_{k=1}^{n} \frac{k}{n^2}$
33. $\sum_{k=1}^{50} \left[(k+1)^2 - k^2\right]$
34. $\sum_{k=2}^{20} \left[\sin(k-1) - \sin k\right]$
35. $\sum_{k=7}^{30} \left(\sqrt{k-4} - \sqrt{k-3}\right)$
36. $\sum_{k=1}^{40} \frac{1}{k(k+1)}$ $\left(\textit{Hint: } \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}\right)$

Riemann Sums

In Exercises 37–42, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

37. $f(x) = x^2 - 1, \quad [0, 2]$
38. $f(x) = -x^2, \quad [0, 1]$
39. $f(x) = \sin x, \quad [-\pi, \pi]$
40. $f(x) = \sin x + 1, \quad [-\pi, \pi]$
41. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.
42. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Riemann Sums

For the functions in Exercises 43–50, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

43. $f(x) = 1 - x^2$ over the interval $[0, 1]$.
44. $f(x) = 2x$ over the interval $[0, 3]$.
45. $f(x) = x^2 + 1$ over the interval $[0, 3]$.
46. $f(x) = 3x^2$ over the interval $[0, 1]$.
47. $f(x) = x + x^2$ over the interval $[0, 1]$.
48. $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.
49. $f(x) = 2x^3$ over the interval $[0, 1]$.
50. $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.

5.3 The Definite Integral

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral
2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution
3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution

$$\boxed{\int_a^b f(t)\,dt}$$

Upper limit of integration

The function $f(x)$ is the integrand.

Integral sign

x is the variable of integration.

$$\boxed{\int_a^b \underbrace{f(x)}\,dx}$$

Lower limit of integration

Integral of f from a to b

When you find the value of the integral, you have evaluated the integral.

"d"

for start

end

$$\int \qquad \sum_{k=1}^{n} f(x_k^*)\ \Delta x$$

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \underbrace{\sum_{k=1}^{n} f(c_k)\,\Delta x_k}_{RS} - J \right| < \varepsilon.$$

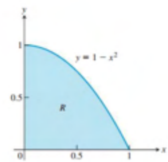
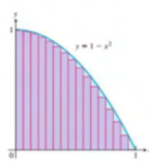


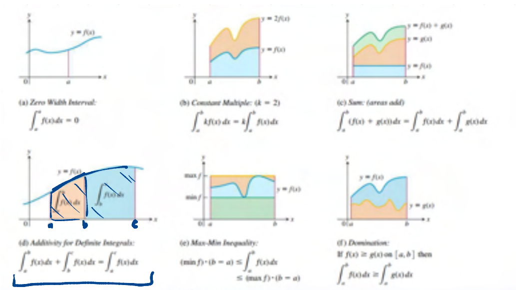
FIGURE 5.1 The area of a region R cannot be found by a simple formula.

A Formula for the Riemann Sum with Equal-Width Subintervals

$$\underbrace{\int_a^b f(x)\,dx}\ =\ \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)}_{\text{right-endpoints}} \qquad (1)$$



(a)



$$\Delta x = \frac{b-a}{n} = -\frac{a-b}{n}$$

TABLE 5.6 Rules satisfied by definite integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A definition when $f(a)$ exists
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6. *Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$.	
7. *Domination:*	If $f(x) \ge g(x)$ on $[a, b]$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$. If $f(x) \ge 0$ on $[a, b]$ then $\int_a^b f(x)\,dx \ge 0$.	Special case

EXAMPLE 2 To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \cancel{\text{and}} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

and $\int_1^4 h(x)\,dx = 3$

Also find

(1)' $\int_{-1}^{4} f(x)\,dx$

1. $\int_4^1 f(x)\,dx = -\underbrace{\int_1^4 f(x)\,dx}_{\text{sub.}} = -(-2) = 2$

(1)' $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_1^4 f(x)\,dx = 5 + (-2) = 3$

2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = \int_{-1}^{1} 2f(x)\,dx + \int_{-1}^{1} 3h(x)\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$

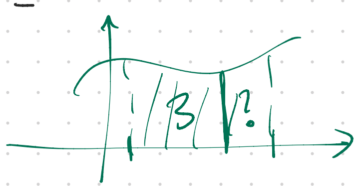
$= 2(5) + 3(7) = 31$

2'. $\int_1^4 f(x)\cdot h(x)\,dx = ?$

we don't know!! $(f(x)\cdot h(x))$?

3. $\int_{-4}^{4} f(x)\,dx =$?? we don't know $(x\text{-value} -4 \,?, -1)$?

3'. $\int_3^4 h(x)\,dx =$



Example 6:

Given that $\int_1^3 2f(x)dx = 4$ and $\int_1^0 f(x)dx = -1$, find $\int_0^3 f(x)dx$.

A. -3
B. 1
C. 3
D. 5

$4 = \int_1^3 2 f(x) dx$
$= 2 \underbrace{\int_1^3 f(x)\, dx}_{?}$

$\int_0^3 f(x)\, dx$
$= \int_0^1 f(x) dx + \int_1^3 f(x) dx$
$= -(-1) + \frac{1}{2}(4)$
$= 1 + 2 = 3$

Even / odd Functions.

Defn.

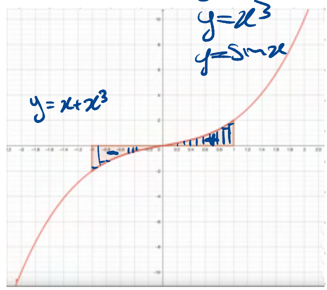
$f(x)$ is even if
$f(-x) = f(x)$.

$f(x)$ is odd if
$f(-x) = -f(x)$

Ex.

$y = x^2$
$y = |x|$
$y = e^{x^2}$
$y = \cos x$

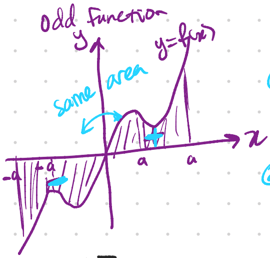
$y = x$
$y = x^3$
$y = \sin x$



Ex. 4 Evaluate the definite integral. (Hint: the functions are odd)

(a) $\int_{-1}^{1} x + x^3 \, dx = 0$

$f(x) = x + x^3$
$f(-x) = -x + (-x)^3$
$= -x - x^3$
$= -(x + x^3) = -f(x)$

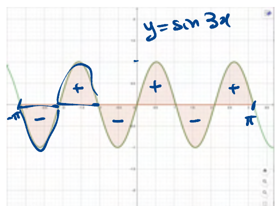


Defn ① $f(x)$ is odd if $f(-x) = -f(x)$ for all x-values.

② Thm If $f(x)$ is odd, then for any number a we have

$$\int_{-a}^{a} f(x)\, dx = 0.$$

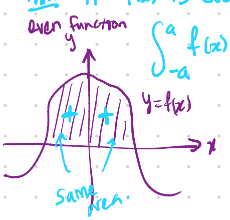
(b) $\int_{-\pi}^{\pi} \sin(3x)\, dx = 0$

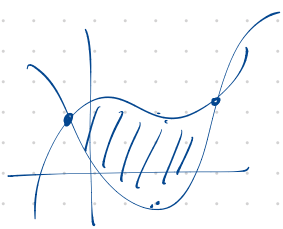


5. $f(x)$ is an even function. If $\int_0^4 f(x)dx = 3$ and $\int_4^6 f(x)dx = 5$, find $\int_{-4}^{6} f(x)dx$.

Thm If $f(x)$ is even, then for any number a

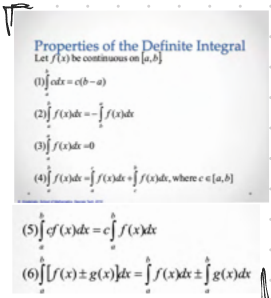
$$\int_{-a}^{a} f(x)\, dx = 2 \cdot \int_0^a f(x)\, dx.$$





$\int_{-4}^{6} f(x)\, dx = \int_{-4}^{0} f(x)\, dx + \int_0^4 f(x)\, dx + \int_4^6 f(x)\, dx$
$= 3 + 3 + 5 = 11$

Some formulas ... if you're into that kind of thing



Properties of the Definite Integral

Let $f(x)$ be continuous on $[a,b]$

(1) $\int_a^b c\,dx = c(b-a)$

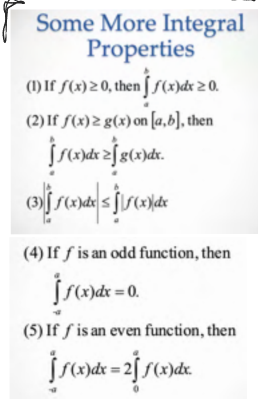
(2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(3) $\int_a^a f(x)dx = 0$

(4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in [a,b]$

(5) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(6) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$



Some More Integral Properties

(1) If $f(x) \geq 0$, then $\int_a^b f(x)dx \geq 0$.

(2) If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

(3) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

(4) If f is an odd function, then $\int_{-a}^{a} f(x)dx = 0$.

(5) If f is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

More examples:

6. $$\int_{-3}^{2} |x|\,dx = \int_{-3}^{2} |x|\,dx + \int_{-2}^{2} |x|\,dx \;(?)$$

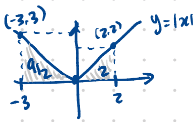
$\int_{-2}^{2} |x|\,dx \rightarrow 2\int_0^2 |x|\,dx$

$$|x| = \begin{cases} -x & x \leq 0 \\ x & x \geq 0 \end{cases}$$

$$= \int_{-3}^{0} |x|\,dx + \int_0^2 |x|\,dx$$

$$= \int_{-3}^{0} -x\,dx + \int_0^2 x\,dx$$

$$= \frac{9}{2} + 2 = \boxed{13/2}$$



7. If $\int_0^2 1+x^2\,dx = 14/3$, find

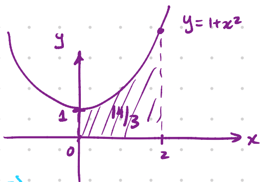
$$\int_{-2}^{2} \underbrace{1+x+x^2}_{f}+x^3\,dx$$

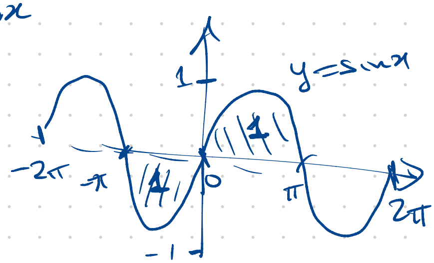
$x(1+x^2)$

$$= \int_{-2}^{2} \overbrace{1+x^2}^{f}\,dx + \int_{-2}^{2} \overbrace{x+x^3}^{g}\,dx$$

$$= 2\int_0^2 1+x^2\,dx + 0$$

$$= 2(14/3) + 0 = \boxed{28/3}$$

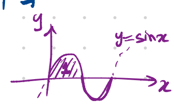




8. If $\int_0^{\pi} \sin(x)dx = 1$, find $\int_{-\pi}^{0} \sin(x)\,dx$.

(Hint: $f(x) = \sin x$ is odd)

$$= \boxed{-1}$$



So now, we can combine the ideas

Find

(1) The anti-derivative

(2) The average value over the interval [a,b]

(3) The definite integral of f(x) over the interval [a,b]

$$\boxed{\int f(x)\,dx = F(x) + C}$$

FTC

avg value of f over [a,b].

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x}$$

FTC II.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

FTC I

$$\Big(\overbrace{\int_a^x f(t)\,dx}^{F(x)}\Big)' = f(x)$$

Use limits of Riemann Sums (not FTC)

Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$

56. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$

57. $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t-1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

(c) $h(x) = \frac{1}{x} + 2e^{2x}$

The function $f(x)$ is the integrand.

Upper limit of integration

Integral sign

x is the variable of integration.

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \varepsilon.$$

A Formula for the Riemann Sum with Equal-Width Subintervals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \qquad (1)$$

EXERCISES 5.3

Using the Definite Integral Rules

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.6 to find

a. $\int_2^2 g(x)\,dx$ **b.** $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ **d.** $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ **f.** $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.6 to find

a. $\int_1^9 -2f(x)\,dx$ **b.** $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ **d.** $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ **f.** $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ **b.** $\int_1^2 \sqrt{3} f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ **d.** $\int_1^2 [-f(x)]\,dx$

5.4 The Fundamental Theorem of Calculus

MATH 1552 COURSE SYLLABUS (IN-PERSON SECTIONS), SUMMER 2023

Tentative Course Schedule

Please use this as an approximate class schedule; section coverage may change depending on the flow of the course. Review days/topics may be changed or cancelled in the event of inclement weather or campus closures.

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral

Today's Learning Goals

- Know the statements of the FTC and the Second FTC
- Apply the 2nd FTC to evaluating definite integrals using the formulas from Section 4.8
- Apply the ~~Second~~ First FTC to differentiate an integral

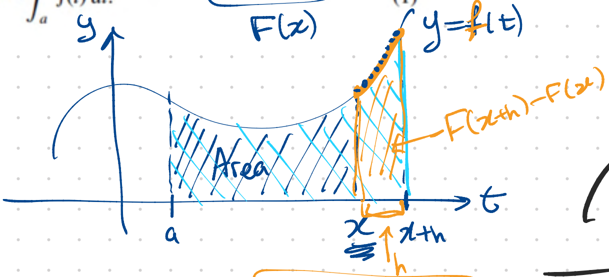
What does it mean?

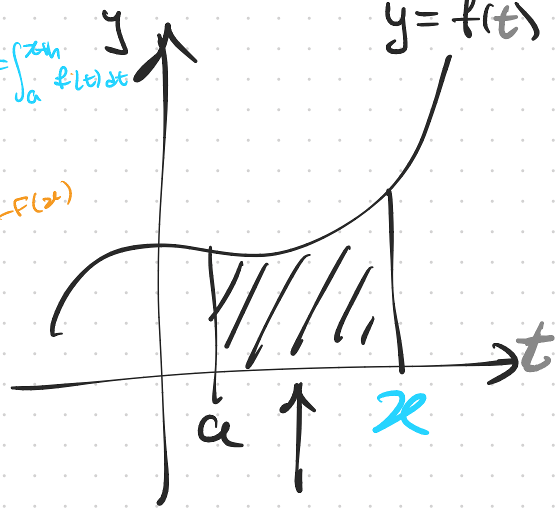
$F'(x) = ??$

$$F(x) = \int_a^x f(t)\, dt. \qquad (1)$$

$$\underbrace{\text{Area} = \int_a^x f(t)\, dt}_{F(x)}$$

$$F(x+h) = \int_a^{x+h} f(t)\, dt$$





$$F'(x) = \lim_{h\to 0} \boxed{\frac{F(x+h)-F(x)}{h}}$$

$$= f(x)$$

Area =

$$\frac{d}{dx}\int_a^x f(t)\, dt = f(x)$$

Announcements:

Week	Mon	Tues	Wed	Thurs	Fri
1	May 15 Introduction to Math 1552 Section 4.8: Anti-derivatives	May 16 Calculus review WS 4.8	May 17 Sections 5.1-5.2: Area under the curve	May 18 WS 5.1 WS 5.2-5.3	May 19 Section 5.3: The Definite Integral
2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution
3	May 29 **NO CLASS** Memorial Day	May 30 WS 5.4 WS 5.5-5.6	May 31 Section 5.6: Area Between Curves	Jun 1 WS 5.5-5.6 cont. WS 5.6 Quiz #2 (5.4-5.6)	Jun 2 Section 8.2: Integration by Parts
4	Jun 5 Section 8.3: Powers of Trig Functions	Jun 6 WS 8.2 WS 8.3	Jun 7 Review for Test 1	Jun 8 **Test #1 (4.8, 5.1-5.6, 8.2-8.3)**	Jun 9 Section 8.4: Trigonometric Substitution

IDEA

If we define

$$F(x) = \int_a^x f(t)\,dt. \qquad (1)$$

Want to show that

$$F'(x) = f(x).$$

This is because the difference quotient

$$\frac{F(x+h) - F(x)}{h} \approx f(x).$$ (for small h)

So in particular

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

or if you prefer...

Some nice pictures

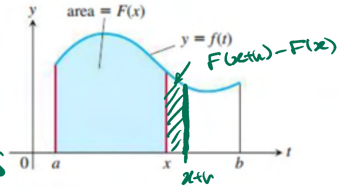


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

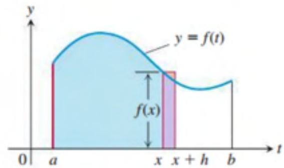


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

Punchline

THM (FTC part 1)

If $F(x) = \int_a^x f(t)\,dt$, then

$$F'(x) = f(x).$$

FTC part 1.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1
If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

EXAMPLE Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\,dt$ (b) $y = \int_x^5 3t \sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$ (d) $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t}\,dt$

(a) If $y = \int_a^x \overbrace{t^3+1}^{f(t)}\,dt$

Then $\frac{dy}{dx} = f(x) = \boxed{x^3+1}$

$$y = \int_a^x t^3+1\,dt$$

$$= \frac{1}{4}t^4 + t \Big|_a^x = \left(\frac{1}{4}x^4 + x\right) - \left(\frac{1}{4}a^4 + a\right)$$

(b) If $y = \int_x^5 3t \sin t\,dt$

Then $\frac{dy}{dt} = -3x \sin x$

If $-y = \int_5^x \overbrace{3t \sin t}^{f(t)}\,dt$

$-\frac{dy}{dt} = 3x \sin x$

$F(x) = \int_a^x f(t)\,dt$

$\Longrightarrow F'(x) = f(x)$

FTC part 1.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1
If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

EXAMPLE Use the Fundamental Theorem to find dy/dx if

✓ **(a)** $y = \int_a^x (t^3 + 1)\,dt$ ✓ **(b)** $y = \int_x^5 3t \sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$ **(d)** $y = \int_{1+3x^2}^4 \frac{1}{2 + e^t}\,dt$

$F(x) = \int_a^x f(t)\,dt$

$\Longrightarrow F'(x) = f(x).$

(c) If $y = \int_1^{x^2} \cos t\,dt$

Then $\frac{dy}{dt} = \underbrace{2x}_{(x^2)'} \cdot \underbrace{\cos \cancel{x^2}}_{f(\cancel{x})}$

why?

Suppose

$F(x) = \int_1^x \cos t\,dt \Rightarrow y = F(x^2) = \int_1^{x^2} \cos t\,dt$

$\Rightarrow F'(x) = f(x)$ ✓

$\Rightarrow \frac{dy}{\cancel{dx}} = F'(x^2) * (x^2)'$

CHAIN.

$= f(x^2) * 2x = \cos(x^2) \cdot 2x$

(d) If $y = \int_{1+3x^2}^4 \overbrace{\frac{1}{2+e^t}}^{f(t)}\,dt$

Then $\frac{dy}{d\cancel{x}} =$

Then $-y = \boxed{\int_4^{1+3x^2} \frac{1}{2+e^t}\,dt}$ $F(1+3x^2)$

and.

$\boxed{F(x) = \int_4^x \frac{1}{2+e^t}\,dt}$

$\Longrightarrow F'(x) = \frac{1}{2+e^x}$

So

$-y = F(1+3x^2)$

$F(\text{✿}) = \int_4^{\text{✿}} \frac{1}{2+e^t}\,dt$

✿-rule.

✿-ran

$-\frac{dy}{dx} = F'(1+3x^2) * (1+3x^2)'$

$= f(1+3x^2)\, 6x$

$= \frac{6x}{2 + e^{1+3x^2}}$

$\frac{dy}{dx} = \frac{-6x}{2 + e^{1+3x^2}}$

Example : Find F′(2).

$$F(x)=\int_1^x \frac{t}{t^3+3}\,dt$$

A. 2/7
B. 2/11
C. ¼
D. 3/44

Step 1: Find $F'(x)$ using FTC.

Step 2 plug in $x=2$.

Step 1: $F'(x) = \dfrac{x}{x^3+3}$

Step 2: $F'(2) = \dfrac{2}{8+3} = 2/11$

If time…

Example : Extension to 2nd FTC

Use this extension:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

to find $F'(x)$ if $F(x) = \int_{3x}^{\cos x} \frac{1}{1+t}\,dt$.

$$= \int_{3x}^{a} \frac{1}{1+t}\,dt + \int_{a}^{\cos x} \frac{1}{1+t}\,dt$$

$$= -\int_{a}^{3x} \frac{1}{1+t}\,dt + \int_{a}^{\cos x} \frac{1}{1+t}\,dt$$

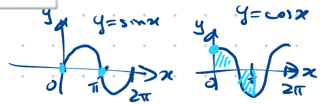
$$= -\frac{1}{1+3x}(3x)' + \frac{1}{1+\cos x}(\cos x)'$$

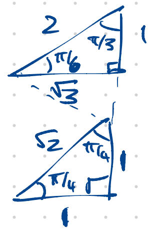
$$= \frac{-3}{1+3x} + \frac{-\sin x}{1+\cos x}$$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

FTC part 2





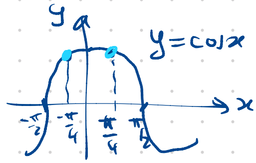
(a) $\displaystyle\int_0^{\pi} \cos x\,dx = \sin x \Big|_0^{\pi}$

$= \sin\pi - \sin 0$

$= 0 - 0 = 0$

$\sec x = \dfrac{1}{\cos x}$

$\cos\theta = \dfrac{adj}{hyp}$



(b) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x \Big|_{-\pi/4}^{0}$

$= \sec 0 - \sec(-\pi/4)$

$= \dfrac{1}{\cos 0} - \dfrac{1}{\cos(-\pi/4)} = \dfrac{1}{1} - \dfrac{1}{1/\sqrt{2}} = 1 - \sqrt{2}$

$\dfrac{1}{\sqrt{2}/2} = \dfrac{2}{\sqrt{2}}$ $\quad \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

(c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \int_1^4 \frac{3}{2}x^{1/2} - 4x^{-2}\,dx$

$= \dfrac{3}{2}\dfrac{x^{1/2+1}}{1/2+1} - 4\dfrac{x^{-1}}{-1} \Big|_1^4$

$= \dfrac{3}{2}\dfrac{x^{3/2}}{3/2} + \dfrac{4}{x}\Big|_1^4$

$= \dfrac{\cancel{3}}{\cancel{2}}\cdot\dfrac{\cancel{2}}{\cancel{3}}x^{3/2} + \dfrac{4}{x}\Big|_1^4 = \left(4^{3/2} + \dfrac{4}{4}\right) - \left(1^{3/2} + \dfrac{4}{1}\right)$

$= 8 + 1 - 1 - 4 = \cancel{12}\ 4$

(d) $\displaystyle\int_0^1 \frac{dx}{x+1} =$

(e) $\displaystyle\int_0^1 \frac{dx}{x^2+1} =$

FTC part 2

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$$(\ln x)' = \frac{1}{x}$$

$$(\ln(x+1))' = \frac{1}{x+1} * \cancel{(x+1)'}^{\,1}$$

$$= \boxed{\frac{1}{x+1}}$$

(d) $\displaystyle\int_0^1 \frac{dx}{x+1} = \ln|x+1| \Big|_0^1$

$$= \ln|1+1| - \ln|0+1|$$

$$= \ln 2 - \cancel{\ln 1}^{\,0}$$

$$= \ln 2$$

$y = e^x$ $\quad e^0 = 1$

$\ln(x) = y$ $\quad \ln(1) = 0$

same thing

(e) $\displaystyle\int_0^1 \frac{dx}{x^2+1} = \tan^{-1}(x) \Big|_0^1$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

is the angle θ such that $\tan(\theta) = 1$

the angle θ s.t. $\tan(\theta) = 0$.

$$= \frac{\pi}{4} - 0 = \pi/4$$

Check your understanding

Example : Evaluate.

$$\int_1^3 \frac{1}{x^2}\,dx$$

A. 2/3
B. 4/3
C. 26/9
D. 26/81

?? $\underbrace{\ln(x^2)}_{F(x)}\Big|_1^3$ $\quad 2\ln(x)$

$$F'(x) \stackrel{?}{=} \frac{1}{x^2}(x^2)' = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$$

$$\int_1^3 \frac{1}{x^2}\,dx = \int_1^3 x^{-2}\,dx$$

$$= \frac{x^{-1}}{-1}\Big|_1^3 = -x^{-1}\Big|_1^3 = \frac{-1}{3} - \left(\frac{-1}{1}\right)$$

$$= \frac{-1}{3} + 1 = 2/3$$

FRIDAY Start here

THEOREM 3—The Mean Value Theorem for Definite Integrals
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Log Base change

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

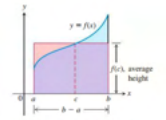


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or mean) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

Example :

The percent of toxin in a lake, where time is in years, is given by the function:

$$f(t) = 50\left(\frac{1}{4}\right)^t.$$

Find the average amount of toxin in the lake between years 1 and 3.

avg value of $f(x)$ over $[a,b]$ $= \frac{1}{b-a}\int_a^b f(x)\,dx$

Soln $\frac{1}{3-1}\int_1^3 f(t)\,dt$ = avg value of $f(t)$ over $[1,3]$

$$= \frac{1}{2}\int_1^3 50\left(\frac{1}{4}\right)^t dt$$

$$= \frac{1}{2}\int_1^3 50 * 4^{-t}\,dt$$

$$= \frac{1}{2}\int_1^3 50\left(e^{\ln(4)}\right)^{-t} dt$$

$$= \frac{1}{2}\int_1^3 50\, e^{(-\ln 4)t}\,dt = 25\int_1^3 e^{-\ln 4\, t}\,dt$$

$$= 25 \cdot \frac{1}{-\ln 4}\, e^{-\ln 4\, t}\Big|_1^3$$

$$= \frac{-25}{\ln 4}\, 4^{-t}\Big|_1^3 = \frac{-25}{\ln 4}\left(4^{-3} - 4^{-1}\right)$$

$$= \frac{-25}{\ln 4}\left(\frac{1}{64} - \frac{1}{4}\right)$$

$$= \frac{15 \cdot 25}{64 \cdot \ln 4}$$

$$= \frac{5^3 \cdot 3}{2^6 \ln 4}$$

$$e^{\ln(a)} = a$$

$$4 = e^{\ln(4)} \checkmark$$

$$\int e^{kt}\,dt = \frac{1}{k}e^{kt} + C$$

$$\left(a^2\right)^3 = a^2 a^2 a^2 = a^6$$

Example 5:

Find the average value of the function:

$$f(x) = 1 - x^2, \ -1 \leq x \leq 3.$$

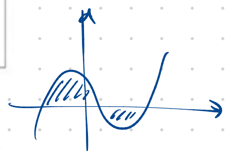
Then find a c that satisfies the MVT for integration.

$$\boxed{\frac{5^3 \cdot 3}{2^6 \cdot \ln 4}}$$

https://strawpoll.com/polls/NoZr3VJMry3

Summary:
To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.



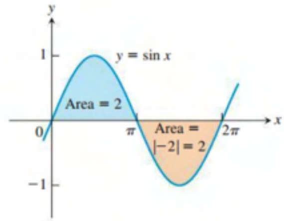


FIGURE 5.22 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

EXAMPLE 8 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

$$\int_{-1}^{2} x^3 - x^2 - 2x \, dx \neq \text{total area}$$

What intervals is
$f(x) > 0$?
$f(x) < 0$?

① Solve

$$f(x) = x^3 - x^2 - 2x = 0$$
$$= x(x^2 - x - 2) = 0$$
$$= x(x-2)(x+1) = 0$$
$$x = 0, 2, -1$$

$f(1) = 1 - 1 - 2 = -1$

$f(-1) = (-1)^3 - (-1)^2 - 2(-1)$
$= -1 - 1 + 2 = 0$

$f(-\frac{1}{2}) = (-\frac{1}{2})^3 - (\frac{1}{2})^2 - 2(-\frac{1}{2})$
$= -\frac{1}{8} - \frac{1}{4} + 1 > 0$

Sign chart

pos f   neg f

$-1 \quad -\frac{1}{2} \quad 0 \quad 1 \quad 2$

$$A_1 = \int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \text{positive area}$$

$$A_2 = \int_{0}^{2} -(x^3 - x^2 - 2x)\, dx = \cancel{\text{negative}}\ \text{positive area}$$

$$A_1 = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \Big|_{-1}^{0} = (0 - 0 - 0) - (\tfrac{1}{4} + \tfrac{1}{3} - 1)$$
$$= 1 - \tfrac{7}{12} = \tfrac{5}{12}$$

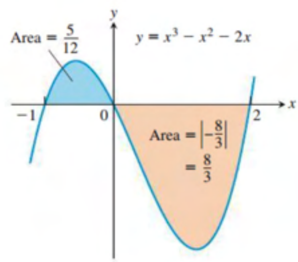


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

$$-A_2 = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \Big|_{0}^{2} = (\tfrac{1}{4}\cdot 16 - \tfrac{1}{3}\cdot 8 - 4) - (0 - 0 - 0)$$
$$= 4 - 8/3 - 4 = -8/3$$

$$A_1 + A_2 = \frac{5}{12} + \frac{8}{3} = \frac{5}{12} + \frac{32}{12} = \boxed{\frac{37}{12}}$$

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–34.

1. $\int_0^2 x(x-3)\,dx$
2. $\int_{-1}^1 (x^2-2x+3)\,dx$
3. $\int_{-2}^2 \frac{3}{(x+3)^4}\,dx$
4. $\int_{-1}^1 x^{299}\,dx$
5. $\int_1^4 \left(3x^2-\frac{x^3}{4}\right)dx$
6. $\int_{-2}^3 (x^3-2x+3)\,dx$
7. $\int_0^1 (x^2+\sqrt{x})\,dx$
8. $\int_1^{32} x^{-6/5}\,dx$
9. $\int_0^{\pi/3} 2\sec^2 x\,dx$
10. $\int_0^{\pi} (1+\cos x)\,dx$
11. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$
12. $\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u}\,du$
13. $\int_{\pi/2}^{0} \frac{1+\cos 2t}{2}\,dt$
14. $\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt$
15. $\int_0^{\pi/4} \tan^2 x\,dx$
16. $\int_{-\pi/4}^{\pi/6} (\sec x+\tan x)^2\,dx$
17. $\int_0^{\pi/8} \sin 2x\,dx$
18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$
19. $\int_1^{-1} (r+1)^2\,dr$
20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\,dt$
21. $\int_{\sqrt{2}}^1 \left(\frac{u^7}{2}-\frac{1}{u^5}\right)du$
22. $\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy$
23. $\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2}\,ds$
24. $\int_1^8 \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}}\,dx$
25. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$
26. $\int_0^{\pi/3} (\cos x+\sec x)^2\,dx$
27. $\int_{-4}^4 |x|\,dx$
28. $\int_0^{\pi} \frac{1}{2}(\cos x+|\cos x|)\,dx$
29. $\int_0^{\ln 2} e^{3x}\,dx$
30. $\int_1^2 \left(\frac{1}{x}-e^{-x}\right)dx$
31. $\int_0^{1/2} \frac{4}{\sqrt{1-x^2}}\,dx$
32. $\int_0^{1/\sqrt{3}} \frac{dx}{1+4x^2}$
33. $\int_2^4 x^{\pi-1}\,dx$
34. $\int_{-1}^0 \pi^{x-1}\,dx$

In Exercises 35–38, guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

35. $\int_0^1 xe^{x^2}\,dx$
36. $\int_1^2 \frac{\ln x}{x}\,dx$
37. $\int_2^5 \frac{x\,dx}{\sqrt{1+x^2}}$
38. $\int_0^{\pi/3} \sin^2 x\cos x\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 39–44.

a. by evaluating the integral and differentiating the result.
b. by differentiating the integral directly.

39. $\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt$
40. $\frac{d}{dx}\int_1^{\sin x} 3t^2\,dt$
41. $\frac{d}{dt}\int_0^{t^4} \sqrt{u}\,du$
42. $\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\,dy$
43. $\frac{d}{dx}\int_0^{x^3} e^{-t}\,dt$
44. $\frac{d}{dt}\int_0^{\sqrt{t}} \left(x^4+\frac{3}{\sqrt{1-x^2}}\right)dx$

Find dy/dx in Exercises 45–56.

45. $y=\int_0^x \sqrt{1+t^2}\,dt$
46. $y=\int_1^x \frac{1}{t}\,dt,\quad x>0$
47. $y=\int_{\sqrt{x}}^0 \sin(t^2)\,dt$
48. $y=x\int_2^{x^2} \sin(t^3)\,dt$
49. $y=\int_{-1}^x \frac{t^2}{t^2+4}\,dt-\int_3^x \frac{t^2}{t^2+4}\,dt$
50. $y=\left(\int_0^x (t^3+1)^{10}\,dt\right)^3$
51. $y=\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}},\quad |x|<\frac{\pi}{2}$
52. $y=\int_{\tan x}^0 \frac{dt}{1+t^2}$
53. $y=\int_0^{e^{x^2}} \frac{1}{\sqrt{t}}\,dt$
54. $y=\int_{2^x}^1 \sqrt[3]{t}\,dt$
55. $y=\int_0^{\sin^{-1}x} \cos t\,dt$
56. $y=\int_{-1}^{x^{1/\pi}} \sin^{-1} t\,dt$

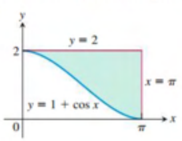
Area

In Exercises 57–60, find the total area between the region and the x-axis.

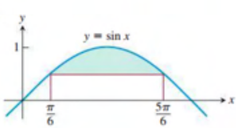
57. $y=-x^2-2x,\quad -3\le x\le 2$
58. $y=3x^2-3,\quad -2\le x\le 2$
59. $y=x^3-3x^2+2x,\quad 0\le x\le 2$
60. $y=x^{1/3}-x,\quad -1\le x\le 8$

Find the areas of the shaded regions in Exercises 61–64.

61.



62.



$\int \tan^2 x\,dx$

$= \int \frac{\sin^2 x}{\cos^2 x}\,dx$

$= \int \frac{1-\cos^2 x}{\cos^2 x}\,dx$

$= \int \sec^2 x - 1\,dx$

5.5 Indefinite Integrals and the Substitution Method

2	May 22 Section 5.3: The Definite Integral cont. Section 5.4: The Fundamental Theorem of Calculus	May 23 WS 5.2-5.3 cont. WS 5.3	May 24 Section 5.4: The Fundamental Theorem of Calculus cont. *Welcome survey and syllabus quiz due!*	May 25 WS 5.3 cont. Quiz #1 (4.8, 5.1-5.3)	May 26 Section 5.5: Integration by Substitution

$\boxed{\text{u- sub}}$

For which of the functions below can we currently find an antiderivative?

X $f(x) = \sec x$

✓ $g(x) = \csc(3x)\cot(3x)$

X $h(x) = x \sin x$

$k(x) = x\cos(x^2)$

u-sub.

Functions we can integrate:

$x^n, \sin(ax), \cos(ax)$

$\csc(ax)\cot(ax)$

$\sec(ax)\tan(ax)$

$\sec^2(ax), \csc^2(ax)$

e^{ax}, b^{ax}

$\dfrac{1}{1+(ax)^2}, \dfrac{1}{\sqrt{1-(ax)^2}}$

$\tan^2 x, \ldots$

THEOREM 6—The Substitution Rule
If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

The Substitution Method to evaluate $\int f(g(x))g'(x)\, dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\, dx = g'(x)\, dx$ to obtain $\int f(u)\, du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

① $$\int x^2 e^{x^3}\, dx = \int e^{x^3} \cdot x^2\, dx$$

$$= \int e^u \cdot \tfrac{1}{3}\, du$$

$$= \int \tfrac{1}{3} e^u\, du$$

$$= \tfrac{1}{3} e^u + C$$

$$= \boxed{\tfrac{1}{3} e^{x^3} + C}$$

u-sub Box

$u = x^3$

$du = 3x^2\, dx$

$\tfrac{1}{3}\, du = x^2\, dx$

$$\int x\sqrt{2x + 1}\, dx.$$

tricky

Hint: multiply by e^x on top and bottom

(a) $\displaystyle\int \frac{dx}{e^x + e^{-x}} =$

Hint: multiply by sec(x)+tan(x) on top and bottom

(b) $\displaystyle\int \sec x\, dx \overset{?}{=} \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$

$$= \int \frac{\overbrace{\sec^2 x + \sec x \tan x}^{du}\, dx}{\underbrace{\sec x + \tan x}_{u}}$$

u-sub Box

$$\boxed{\begin{aligned} u &= \sec x + \tan x \\ du &= \sec x \tan x + \sec^2 x \, dx \end{aligned}}$$

$$= \int \frac{1}{u}\, du$$

$$= \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

how to check ans??

Integrals of the tangent, cotangent, secant, and cosecant functions

$\int \tan x\, dx = \ln|\sec x| + C$ $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

$\int \cot x\, dx = \ln|\sin x| + C$ $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

$$\left(\ln|\sec x + \tan x| + C\right)'$$

$$= \left(\frac{1}{\sec x + \tan x} + 0\right) * (\sec x + \tan x)'$$

$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

2

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{1}{\sqrt[3]{\underbrace{z^2+1}_{u}}} \cdot \underbrace{2z\, dz}_{du}$$

Method 1: Substitute $u = z^2 + 1$.

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

u-sub box

$$\boxed{\begin{aligned} u &= z^2 + 1 \\ du &= 2z\, dz \end{aligned}}$$

$$= \int \frac{1}{\sqrt[3]{u}}\, du$$

$$= \int u^{-1/3}\, du$$

$$= \frac{u^{-1/3+1}}{-1/3+1} + C$$

$$= \frac{u^{2/3}}{2/3} + C = \boxed{\frac{3}{2}(z^2+1)^{2/3} + C}$$

$\sin^2(x) = 1 - \cos^2 x$

$$(a)\int \frac{\cos(\sqrt{t})}{\sqrt{t}\,\sin^2(\sqrt{t})}\,dt = \int \frac{\cos(u)}{\sin^2(u)}\, 2\,du = \int \frac{\cos u}{1-\cos^2 u}\cdot 2\,du \;??$$

$$\int \frac{1}{w^2}\, 2\cdot dw$$

u-sub Box

$u = \sqrt{t}$
$du = \frac{1}{2\sqrt{t}}\,dt$
$2\,du = \frac{1}{\sqrt{t}}\,dt$

$(\sqrt{t})' = (t^{1/2})'$
$= \frac{1}{2}t^{1/2-1}$
$= \frac{1}{2}t^{-1/2}$
$= \frac{1}{2}\cdot\frac{1}{\sqrt{t}}$

u-sub

$u = \sin\sqrt{t}$
$du = \cos(\sqrt{t})\cdot\frac{1}{2\sqrt{t}}\,dt$

w-sub

$w = \sin^2(u)$
$dw = 2\sin(u)\cos(u)\,du$

w-sub

$w = \sin u$
$dw = \cos u\,du$

$$= \int 2w^{-2}\,dw = -2w^{-1} + C$$
$$= -2(\sin u)^{-1} + C$$
$$= -2(\sin\sqrt{t})^{-1} + C$$
$$= \boxed{\frac{-2}{\sin\sqrt{t}} + C}$$

$$(b)\int_2^e \frac{1}{x(\ln x)^3}\,dx = \int_2^e \frac{1}{(\ln x)^3}\cdot\underbrace{\frac{1}{x}\,dx}_{du} \;??$$

u-sub Box

$u = \ln x$
$du = \frac{1}{x}\,dx$

$$= \int_*^* \frac{1}{u^3}\cdot du$$
$$= \int_*^* u^{-3}\,du = \frac{-1}{2}u^{-2}\Big|_*^*$$
$$= \frac{-1}{2}(\ln(x))^{-2}\Big|_2^e$$
$$= \frac{-1}{2}(\ln e)^{-2} + \cancel{C} - \left(\frac{-1}{2}(\ln 2)^{-2} + \cancel{C}\right)$$
$$= \frac{-1}{2}(1)^{-2} + \frac{1}{2}\cdot\frac{1}{(\ln 2)^2} = \boxed{\frac{1}{2(\ln 2)^2} - \frac{1}{2}}$$

$\int 2x\,dx = x^2 + C$

$$(c)\int w\sqrt{1+w}\,dw$$

EXERCISES 5.5

Evaluating Indefinite Integrals

Evaluate the indefinite integrals in Exercises 1–16 by using the given substitutions to reduce the integrals to standard form.

1. $\int 2(2x+4)^5\,dx,\quad u = 2x+4$

2. $\int 7\sqrt{7x-1}\,dx,\quad u = 7x-1$

3. $\int 2x(x^2+5)^{-4}\,dx,\quad u = x^2+5$

4. $\int \frac{4x^3}{(x^4+1)^2}\,dx,\quad u = x^4+1$

5. $\int (3x+2)(3x^2+4x)^4\,dx,\quad u = 3x^2+4x$

6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx,\quad u = 1+\sqrt{x}$

7. $\int \sin 3x\,dx,\quad u = 3x$

8. $\int x\sin(2x^2)\,dx,\quad u = 2x^2$

9. $\int \sec 2t\tan 2t\,dt,\quad u = 2t$

10. $\int \left(1-\cos\frac{t}{2}\right)^2\sin\frac{t}{2}\,dt,\quad u = 1-\cos\frac{t}{2}$

11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}},\quad u = 1-r^3$

12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy,\quad u = y^4+4y^2+1$

13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx,\quad u = x^{3/2}-1$

14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx,\quad u = -\frac{1}{x}$

15. $\int \csc^2 2\theta\cot 2\theta\,d\theta$

 a. Using $u = \cot 2\theta$
 b. Using $u = \csc 2\theta$

16. $\int \frac{dx}{\sqrt{5x+8}}$

 a. Using $u = 5x+8$
 b. Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 17–66.

17. $\int \sqrt{3-2s}\,ds$

18. $\int \frac{1}{\sqrt{5s+4}}\,ds$

19. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

20. $\int 3y\sqrt{7-3y^2}\,dy$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\int \sqrt{\sin x}\cos^3 x\,dx$

23. $\int \sec^2(3x+2)\,dx$

24. $\int \tan^2 x\sec^2 x\,dx$

25. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

26. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

27. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

28. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

29. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

30. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

31. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

32. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

33. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

34. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

35. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

36. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

37. $\int \frac{x}{\sqrt{1+x}}\,dx$

38. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

39. $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

41. $\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx$

42. $\int \sqrt{\frac{x^4}{x^3-1}}\,dx$

43. $\int x(x-1)^{10}\,dx$

44. $\int x\sqrt{4-x}\,dx$

45. $\int (x+1)^2(1-x)^5\,dx$

46. $\int (x+5)(x-5)^{1/3}\,dx$

47. $\int x^3\sqrt{x^2+1}\,dx$

48. $\int 3x^5\sqrt{x^3+1}\,dx$

49. $\int \frac{x}{(x^2-4)^3}\,dx$

50. $\int \frac{x}{(2x-1)^{2/3}}\,dx$

51. $\int (\cos x)e^{\sin x}\,dx$

52. $\int (\sin 2\theta)e^{\sin^2\theta}\,d\theta$

53. $\int \frac{1}{\sqrt{x}e^{-\sqrt{x}}}\sec^2(e^{\sqrt{x}}+1)\,dx$

54. $\int \frac{1}{x^2}e^{1/x}\sec(1+e^{1/x})\tan(1+e^{1/x})\,dx$

55. $\int \frac{dx}{x\ln x}$

56. $\int \frac{\ln\sqrt{t}}{t}\,dt$

57. $\int \frac{dz}{1+e^z}$

58. $\int \frac{dx}{x\sqrt{x^4-1}}$

59. $\int \frac{5}{9+4r^2}\,dr$

60. $\int \frac{1}{\sqrt{e^{2\theta}-1}}\,d\theta$