Problem 1 Definitions

(20 points)

(i) Define what it means for the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ to be linearly independent.

(ii) What does it mean for a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ to be linear?

(iii) What does it mean for a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ to be onto.

(iv) What is a basis for a subspace V of $\mathbb{R}^n$.

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Problem 2 True or False

No need to justify. In the following A, B, and C are matrices and I stands for the identity matrix. (20 points)

(i) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are linearly independent, then one of them is a multiple of the others.

(ii) If A and B are square matrices then $AB = BA$.

(iii) If $AB = AC$ and A is invertible, then $B = C$.

(iv) An $m \times n$ matrix A is invertible if its columns are linearly independent.

(v) A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with standard matrix A is onto if its columns span $\mathbb{R}^n$.

(vi) If A is $m \times n$ and B is $n \times p$, then the transpose of AB is $A^T B^T$.

(vii) If A is a 3×3 matrix and the equation $A\vec{x} = (1, 1, 1)$ has a unique solution, then A must be invertible.

(viii) If A is 3×4, then A cannot be one-to-one.

(ix) If A is $m \times n$ and the dimension of $ColA = m$, then $A\vec{x} = \vec{b}$ is always consistent.

(x) The solution set to $A\vec{x} = 0$ is a span.

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Problem 3

(i) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first rotates a vector $\vec{x}$ by $90°$ in the **clockwise direction** and then reflects with respect to the first bisector $y = x$. Write down the standard matrix A of T.

(ii) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects with respect to the first bisector $y = x$ and then rotates by angle $90°$ in the **counter-clockwise** direction. Write down the standard matrix B of S.

(iii) How are A and B related? Choose one of the following:

a) A and B are transposes of each other.

b) They have the same columns interchanged.

c) A and B are inverses of each other.

d) The column space of A is the same as the nullspace of B.

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Problem 4

Consider the matrix

$$A = \begin{bmatrix} 0 & -3 & 6 \\ 2 & 1 & -8 \\ -1 & 4 & -5 \\ 1 & -4 & 5 \end{bmatrix}$$

(i) Is A onto? Justify.

(ii) Is it one-to-one? Justify.

(iii) Find the basis for the $Col(A)$.

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Problem 5

Let A be the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 2 & -4 & -1 \end{bmatrix}$$

(i) Find the inverse of the following matrix A if it exists.

(ii) Use the answer of part i) to solve the equation $A\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ without performing any further row reductions.

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