A

1. Let A be the matrix

$$A = \begin{bmatrix} 4 & 8 & 2 & 0 \\ 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 4 \\ 2 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find Nul(A) the null space of A. Be specific in your answer. (10 pts.)

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boxed{\text{nul}(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}}$$

(b) Find a basis for Col(A) the column space of A. (6 pts.)

$$\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}\right\}$$

(c) Describe Nul(A) geometrically in a few words. (2 pts.)

a line in $\mathbb{R}^4$

(d) Describe Col(A) geometrically in a few words. (2 pts.)

a 3-plane in $\mathbb{R}^5$

2. Let $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

(a) Find A^{-1}. (8 pts.)

$$A^{-1} = \frac{1}{4\cdot 3 - 1\cdot 2}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} .3 & -.1 \\ -.2 & .4 \end{bmatrix}}$$

(b) Find the coordinate vector $\mathbf{x}$ of $v = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ in the basis $\left\{ \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

Hint: these are the columns of A. (8 pts.)

Soln x s.t.

$$Ax = b$$

$$x = A^{-1}b$$

$$x = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -6-5 \\ 4+20 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} -11/10 \\ 12/5 \end{bmatrix}}$$

(c) Does $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ belong to the column space of A for any $u_1, u_2 \in \mathbb{R}$? Justify your answer for full credit. (4 pts.)

$\boxed{\text{Yes.}}$ Since A is invertible $Ax = b$ always has a unique soln for any b (so it has "a solution").

3. Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Find A^{-1}. (12 pts.)

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

det * 2 * 2

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -\cancel{0} & -\cancel{0} & 2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] = [I \mid A^{-1}]$$

det * -1

So $\boxed{A^{-1} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ -1 & -1 & 2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}}$

(b) Find det(A). (10 pts.)

det(A) = −4 using row operations.

(c) Check your answer to part *(a)* by appealing to the definition of inverse. (4 pts.)

$$\frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ -2 & -2 & 4 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^{-1} \cdot A = I$

4. Find a basis for (8 pts.)

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}2\\0\\0\\2\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\begin{bmatrix}3\\3\\3\\3\end{bmatrix}\right\}.$$

basis is $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$

5. If a 6×4 matrix A has exactly 3 pivot positions, then the null space Nul(A) is a subspace of $\mathbb{R}^k$ and the column space Col(A) is a subspace of $\mathbb{R}^\ell$. In this problem you will specify the values of k, ℓ, and state the rank and nullity of A. (2 pts. each)

(a) Nul(A) is a subspace of $\mathbb{R}^k$. Specify the value of k.

$k = 4$

(b) Col(A) is a subspace of $\mathbb{R}^\ell$. Specify the value of ℓ.

$\ell = 6$

(c) What is the dimension of Nul(A) the null space of A?

$\dim \text{Nul}(A) = 1$

(d) What is the rank of A?

$\text{rank}(A) = 3$

6. Suppose A is a 2×2 matrix and the null space $\text{Nul}(A)$ is the line in $\mathbb{R}^2$ given by the equation $y = 3x$.

(a) What is $\det(A)$? Justify your answer for full credit. (4 pts.)

$\boxed{\det(A) = 0}$ since $\text{nul}(A) \neq \{0\}$.

(b) What is the rank of A? Justify your answer for full credit. (4 pts.)

$\boxed{\text{rank}(A) = 1}$ since $\dim \text{nul}(A) + \text{rank}(A) = \#\text{ cols}$

$1 + \text{rank}(A) = 2.$

7. Suppose A and B are square 2×2 matrices and you can assume that A, B, and $A + B$ are all invertible, and that $AB = BA$. Find the matrix equal to the following expression, that is, simplify the following expression. (5 pts.)

$$(A + B)^{-1}[(A^2 - B^2)A^{-1} - (A + B)]A$$

$$(A+B)^{-1}\left[(A+B)(A-B)A^{-1}A - (A+B)A\right]$$

$$= A - B - (A+B)^{-1}(A+B)A$$

$$= A - B + A = \boxed{-B.}$$

8. Suppose A is any 3×3 matrix such that the ref of A is $A \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$. Is it true that a basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \right\}$? Either give a counter-example or justify your answer in some way for full credit. (5 pts.)

Yes. $\text{rank}(A) = 3$ so $\text{col}(A) = \mathbb{R}^3$

and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \right\}$

is a basis of $\mathbb{R}^3$.