

1. Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 & 0 \\ 6 & 0 & 4 & 0 \\ 3 & 0 & 1 & -2 \\ 6 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

B

(a) Find $\text{Nul}(A)$ the null space of A . Be specific in your answer.

(10 pts.)

$$A = \begin{bmatrix} 3 & 1 & 2 & 0 \\ 6 & 0 & 4 & 0 \\ 3 & 0 & 1 & -2 \\ 6 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 3 & 0 & 1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & -4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = s \begin{bmatrix} 4/3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$\text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} 4/3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

(6 pts.)

(b) Find a basis for $\text{Col}(A)$ the column space of A .

$$\left\{ \begin{bmatrix} 3 \\ 6 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

(c) Describe $\text{Nul}(A)$ geometrically in a few words.

(2 pts.)

a line in \mathbb{R}^4

(d) Describe $\text{Col}(A)$ geometrically in a few words.

(2 pts.)

a 3-plane in \mathbb{R}^5

2. Let $A = \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix}$.

(a) Find A^{-1} .

(8 pts.)

$$A^{-1} = \frac{1}{3 \cdot 2 - 2 \cdot 4} \begin{bmatrix} 2 & -2 \\ -4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 2 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -3/2 \end{bmatrix}$$

(b) Find the coordinate vector x of $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in the basis $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$.

Hint: these are the columns of A .

(8 pts.)

Solve to

$$Ax = v$$

is

$$x = A^{-1}v$$

So

$$x = \begin{bmatrix} -1 & 1 \\ 2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 + 2 \\ 6 - 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(c) Does $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ belong to the column space of A for any $u_1, u_2 \in \mathbb{R}$? Justify your answer for full credit.

(4 pts.)

yes. Since A is invertible

$$\text{col}(A) = \mathbb{R}^2 \text{ so}$$

$Ax = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is never inconsistent.

3. Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Find A^{-1} .

(12 pts.)

$$\begin{aligned} [A|I] &= \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\det \times 2 \times 2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array} \right] = [I|A^{-1}] \end{aligned}$$

So $A^{-1} = \begin{bmatrix} 0 & -1/2 & -1/2 \\ -1 & -1 & 2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$

(b) Find $\det(A)$.

(10 pts.)

$\det(A) = -4$ using row operations,

(c) Check your answer to part (a) by appealing to the definition of inverse. (4 pts.)

$$\frac{1}{2} \begin{bmatrix} 2 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ -2 & -2 & 4 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1} \cdot A = I \checkmark$$

4. Find a basis for

(8 pts.)

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right\}.$$

$$2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

basis
is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

5. If a 5×7 matrix A has exactly 4 pivot positions, then the null space $\text{Nul}(A)$ is a subspace of \mathbb{R}^k and the column space $\text{Col}(A)$ is a subspace of \mathbb{R}^ℓ . In this problem you will specify the values of k , ℓ , and state the rank and nullity of A . (2 pts. each)

(a) $\text{Nul}(A)$ is a subspace of \mathbb{R}^k . Specify the value of k .

$$k = 7$$

(b) $\text{Col}(A)$ is a subspace of \mathbb{R}^ℓ . Specify the value of ℓ .

$$\ell = 5$$

(c) What is the dimension of $\text{Nul}(A)$ the null space of A ?

$$\dim \text{nul}(A) = 7 - 4 = 3$$

(d) What is the rank of A ?

$$\text{rank } A = 4$$

6. Suppose A is a 2×2 matrix and the null space $\text{Nul}(A)$ is the line in \mathbb{R}^2 given by the equation $y = 5x$.

(a) What is $\det(A)$? Justify your answer for full credit. (4 pts.)

$$\boxed{\det(A) = 0} \text{ since } \text{nul}(A) \neq \{0\}.$$

(b) What is the rank of A ? Justify your answer for full credit. (4 pts.)

$$\boxed{\text{rank } A = 1} \text{ since}$$

$$\dim \text{nul}(A) + \text{rank}(A) = \# \text{ cols}$$

$$1 + \text{rank}(A) = 2.$$

7. Suppose A and B are square 2×2 matrices and you can assume that A , B , and $A + B$ are all invertible, and that $AB = BA$. Find the matrix equal to the following expression, that is, simplify the following expression. (5 pts.)

$$(A + B)^{-1}[(A^2 - B^2)B^{-1} + (A + B)]B$$

$$(A + B)^{-1} \left[(A + B)(A - B)B^{-1} \cdot B + (A + B)B \right]$$

$$= (A + B)^{-1} (A + B) \cdot (A - B) + (A + B)^{-1} (A + B) B$$

$$= A - B + B = \boxed{A}$$

8. Suppose A is any 3×3 matrix such that the ref of A is $A \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Is it true that

a basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$? Either give a counter-example or justify your answer in some way for full credit. (5 pts.)

Yes. Since $\text{rank}(A) = 3$, $\text{Col}(A) = \mathbb{R}^3$

and

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for \mathbb{R}^3 .