Additional Problems for Midterm 1 Review

About This Review Set

As stated in the syllabus, a goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, solutions are not are provided for the additional review problem sets. This is intentional: upper level courses often don't have recitations, let alone worksheets and worksheet solutions. So students need, develop, and use various strategies to check their solutions in those courses. In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses.

The MML Study Plan

If you would like to prepare for the midterm by solving problems that have solutions, the MML Study Plan has hundreds of problems you can solve. MML will tell you if your work is correct and offers a few different study aids. To access problems for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your Math 1553 course
3. select Lay Linear Algebra (the online textbook)
4. select a chapter
5. select a section
6. click study plan

Questions

1. True of False? Circle your answer on the left. Explain briefly in one sentence, possibly by providing a counterexample.

   True **False** A linear transformation with domain $\mathbb{R}^2$ and codomain $\mathbb{R}^3$ has a 2×3 standard matrix.

   True **False** A 3×4 matrix may have 4 pivot positions.

   True **False** If A has a pivot position in every column, then its columns are linearly independent.

   True **False** If a system of equations has more variables than equations, then it has infinitely many solutions.

   True **False** In a set of linearly dependent vectors, every vector can be expressed as a linear combination of the other vectors.

2. Circle TRUE if the statement is *always* true, otherwise circle FALSE. **Justify your answers with a short explanation or counterexample**. Unless otherwise stated, A is a nonzero $m \times n$ matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^m$.

   (a) If the columns of A are linearly dependent, then $A\mathbf{x} = \mathbf{0}$ has a unique solution. TRUE FALSE

   (b) If $m = n$, then $A\mathbf{x} = \mathbf{b}$ has a unique solution. TRUE FALSE

   (c) If $m < n$, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. TRUE FALSE

   (d) If $m > n$, then the columns of A are linearly independent. TRUE FALSE

(e) The matrix $\begin{pmatrix} 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ is in reduced row echelon form. TRUE FALSE

3. Express all solutions to the linear system in parametric vector form.

$$\begin{array}{rrrl} x_1 & +3x_2 & & = 7 \\ 3x_1 & +9x_2 & -5x_3 & = 6 \end{array}$$

4. A polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ takes the values $p(-1) = 3$, $p(1) = 4$ and $p(2) = 0$. Construct an augmented matrix that can be used to compute the coefficients of the polynomial.

5. If possible, fill in the missing entries of the matrices with numbers so that their columns are linearly dependent.

$$A = \begin{bmatrix} 0 & 3 & -9 \\ 1 & 0 & 3 \\ 4 & -10 & \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & \\ 0 & & \end{bmatrix}$$

6. We often have one of two possible goals for applying row reduction algorithms: express a system in echelon form, or in reduced echelon form. And we have so far explored many different kinds of questions that are related the row reduction algorithm. You will need to know whether you need to express a system in echelon, or reduced echelon form.

For the questions below, A is a $m \times n$ matrix, and T_A is the linear transformation associated to A. For each question choose whether the echelon form (EF) or the reduced echelon form (REF) is needed.

EF **REF** Identify a solution to $A\vec{x} = \vec{b}$.

EF **REF** Is the system $A\vec{x} = \vec{b}$ consisistent?

EF **REF** Which columns in the matrix A are pivotal/free?

EF **REF** Determine whether the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent.

EF **REF** For what values of h is $\vec{y}$ in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$?

EF **REF** Compute the coefficients of the polynomial that passes through the following points.

EF **REF** Express the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form.

EF **REF** Determine whether the linear transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ is one-to-one.

EF **REF** Is the linear transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ onto?

7. If possible, give one example of the following.

(a) A 4×2 non-zero matrix A, that is in reduced echelon form, such that $A\vec{x} = \vec{0}$ has a non-trivial solution.

(b) A 3×7 matrix A, in reduced echelon form,with 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 1 free variable.

(c) A 3×7 matrix A, in reduced echelon form,with 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 5 free variables.

(d) A homogeneous linear system that has no solutions.

(e) A homogeneous linear system with two equations and two unknowns that has only one solution.

(f) A is a 3×3 matrix in reduced echelon form, has linearly dependent columns, and exactly two pivot columns.

(g) A is a 4×3 matrix, with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Vectors $\vec{a}_1$ and $\vec{a}_2$ are linearly independent, and $\vec{a}_3 \in \text{Span}\{\vec{a}_1, \vec{a}_2\}$.

8. In each case, give an example of a matrix A in reduced echelon form such that the following holds for the linear transformation $T_A(\vec{x}) = A\vec{x}$:

   (a) $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

   (b) $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

9. Matrix A has dimensions 5×6, and has exactly 2 pivot columns.

   (a) For the system $A\vec{x} = \vec{b}$, how many free variables would there be?

   (b) True or False: $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$.

10. Explain the following in one sentence.

    (a) What it means for a vector $\vec{v}$ to be in the span of vectors $\vec{a}_1$ and $\vec{a}_2$. Vectors $\vec{a}_1$ and $\vec{a}_2$ may or may not be linearly independent.

    (b) What it means, geometrically, for two vectors, $\vec{u}$ and $\vec{v}$, to be linearly dependent.

    (c) What it means, geometrically, for a linear system in three variables and three equations to have an infinite number of solutions.

11. Consider the following linear transformation, and answer **briefly** the questions below.

$$T(x_1, x_2) = \left(\frac{1}{2}x_1 - \frac{1}{2}x_2, -\frac{1}{2}x_1 + \frac{1}{2}x_2\right)$$

    (a) What is the domain of T?

    (b) What is the codomain of T?

    (c) What is the standard matrix of T?

    (d) What are the images of the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$?

    (e) Draw an accurate sketch of $\mathbf{e}_1$, $\mathbf{e}_2$, $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ on one picture.

    (f) Is T onto? Why?

    (g) Is T one-to-one? Why?

    (h) Geometrically describe the range of T.

12. Determine if the following sets of vectors span a point, a line, a plane or all of $\mathbb{R}^3$. Explain your answer without doing any row reductions.

    (a)

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \\ 20 \end{bmatrix}$$

    (b)

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

13. Write the solution for the following system of equations in parametric vector form.

$$\begin{aligned} 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 1 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= -1 \\ x_1 - 2x_2 - x_3 + 3x_4 &= 0 \end{aligned}$$

14. Find all k such that the following vectors are linearly *independent*.

$$\begin{bmatrix} k \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

15. Let A be a matrix that is row equivalent to the given matrix:

$$\begin{pmatrix} 1 & 3 & 0 & -4 \\ 0 & -3 & 6 & -1 \\ 2 & 6 & 0 & -8 \end{pmatrix}$$

(a) Find the reduced row echelon form of the augmented matrix $(A|\mathbf{0})$.

(b) Write the set of solutions to $A\mathbf{x} = \mathbf{0}$ in parametric form.

16. Say whether each of the following functions is linear or not. If it is linear, find the standard matrix. If not, give an example that shows why not.

(a) The function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2z + y \\ y - z \\ 2x + 3z \end{pmatrix}$$

(b) The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points in the counterclockwise direction by $\pi/3$ and then shifts points in the vertical direction by 2.

17. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(a) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent?

(b) Give a geometric description of Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(c) Write $\mathbf{u} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

18. Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Also let $T : \mathbb{R}^a \to \mathbb{R}^b$ be the linear transformation that satisfies

$$T(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \quad T(\mathbf{e}_3) = \begin{pmatrix} -3 \\ -6 \\ 2 \\ 4 \end{pmatrix}.$$

(a) What are a and b?

(b) What is $T(\mathbf{u})$, where $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$?

(c) Is T onto?

(d) Is T one to one?