1. Solve the system of linear equations. (12 pts.)

$$\begin{aligned} -x - y \quad + z &= -1 \\ 3x + y \quad &= 7 \\ -5x - 2y \quad &= -11 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 3 & 1 & 0 & 7 \\ -5 & -2 & 0 & -11 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & -1 & 1 & -1 \\ 0 & -2 & 3 & 4 \\ 0 & 3 & -5 & -6 \end{array}\right] \underset{R_2+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 4 \\ 0 & 1 & -2 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} x &= 3 \\ y &= -2 \\ z &= 0 \end{aligned}}$$

Check
$-3 + 2 + 0 = -1$ ✓
$9 - 2 = 7$ ✓
$-15 + 4 = -11$ ✓

works ✓

2. Is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} \right\}$? Justify your answer for full credit. (12 pts.)

$Ax = b$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & 1 \\ 3 & 9 & 0 & 5 & 1 \\ 1 & 3 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & 1 \\ 0 & 0 & 0 & -10 & -2 \\ 0 & 0 & 0 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 1 & -1/4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 & 1/5 \\ 0 & 0 & 0 & 0 & -1/5 - 1/4 \end{array}\right]$$

$\uparrow \neq 0$

$Ax = b$ inconsistent

so b not in the span of cols of A.

$\boxed{\text{No}}$

3. Find the standard matrix of the linear transformation (10 pts.)

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ y+z \\ x+2y \end{bmatrix}$$

$A = [T(e_1)\ T(e_2)\ T(e_3)]$

$T(e_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix}$

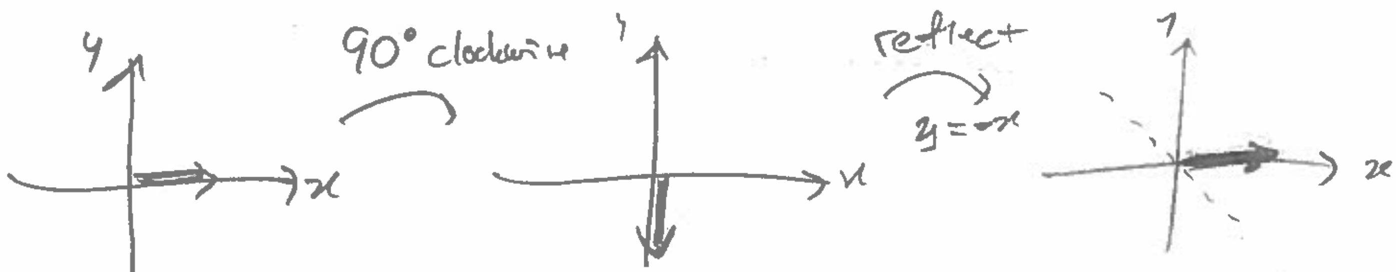
$T(e_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\\2\end{bmatrix}$

$T(e_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$

So $\boxed{A = \begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 1\\ 1 & 2 & 0\end{bmatrix}}$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which first rotates a vector in $\mathbb{R}^2$ by $90°$ **clockwise**, then reflects the resulting vector across the line $y = -x$. Find the standard matrix of T. (10 pts.)

For e_1



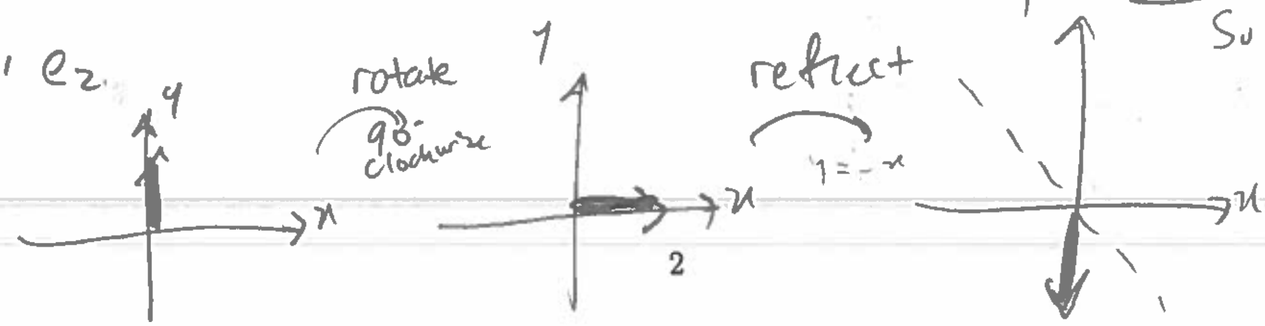
So $T(e_1) = e_1 = \begin{bmatrix}1\\0\end{bmatrix}$

$A = [T(e_1)\ T(e_2)]$

$\boxed{A = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}$

So $T(e_2) = \begin{bmatrix}0\\-1\end{bmatrix}$

For e_2



5. Suppose A is row equivalent to

$$A \sim \begin{array}{c} \begin{matrix} x & y & z & u & v \end{matrix} \\ \begin{bmatrix} 1 & 0 & -2 & 8 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 2 & -8 & 1 \end{bmatrix} \end{array}.$$

Write the solutions of $A\mathbf{x} = 0$ in **parametric vector form.** (10 pts.)

$$A \sim \begin{array}{c} \begin{matrix} & & r & s & \end{matrix} \\ \begin{bmatrix} 1 & 0 & -2 & 8 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$x = 2r - 8s$
$y = -s$
$z = r$ (free
$u = s$ (free)
$v = 0$

$x - 2r + 8s = 0$
$y + s = 0$
$v = 0$
$z = r$ (free)
$u = s$ (free)

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} 2r - 8s \\ -s \\ r \\ s \\ 0 \end{bmatrix} = \boxed{r \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

6. Give an example of a 3×4 matrix with ~~3 rows and 4 columns~~ whose columns span $\mathbb{R}^3$. You must clearly justify that your answer is correct or explain why this is not possible in a few words for full credit. (10 pts.)

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ works.}}$$

(any answer w/ 3 pivots ok if you show me there are EXACTLY 3 pivots)

7. Are the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$ linearly independent or linearly dependent? Fully justify your answer for full credit. (10 pts.)

$$\begin{bmatrix}1&1&1\\0&2&0\\1&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&2&0\\1&1&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

Since there are 3 pivots,

the three vectors are $\boxed{\text{linearly ind.}}$

8. Give an example of three vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{b}$ in $\mathbb{R}^4$ such that $\mathbf{b}$ is not in span$\{\mathbf{v}, \mathbf{w}\}$ but the set $\{\mathbf{v}, \mathbf{w}, \mathbf{b}\}$ is a linearly dependent set of vectors. Your answer must be clearly justified. (8 pts.)

$$\boxed{v = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad w = \begin{bmatrix}2\\2\\2\\2\end{bmatrix} \quad b = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}}$$

$b \notin \text{span}\{v, w\}$

but

$\{b, w, v\}$ <u>lin dep</u>

(many answers possible but need $v = c \cdot w$)

9. For each 3×3 matrix below, determine if the matrix is in row reduced echelon form (RREF) or not. In each case, if the matrix is RREF circle the pivots, and if it is not then explicitly explain which property of RREF is being violated. (3 pts. each)

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ need to clear this 1.

RREF/NOT RREF

(b) $\begin{bmatrix} \cancel{4} & 2 & 3 \\ 0 & \cancel{2} & \cancel{3} \\ 0 & 0 & 3 \end{bmatrix}$ these leading entries need to be 1.

RREF/NOT RREF

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ yes

RREF/NOT RREF

(d) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ No. Not a "staircase"

RREF/NOT RREF

(e) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ yes

RREF/NOT RREF

(f) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ yes

RREF/NOT RREF