Additional Problems for Midterm 2 Review

About This Review Set

As stated in the syllabus, a goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, solutions are not are provided for the additional review problem sets. This is intentional: upper level courses often don't have recitations, let alone worksheets and worksheet solutions. So students need, develop, and use various strategies to check their solutions in those courses. In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses.

The MML Study Plan

If you would like to prepare for the midterm by solving problems that have solutions, the MML Study Plan has hundreds of problems you can solve. MML will tell you if your work is correct and offers a few different study aids. To access problems for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your Math 1553 course
3. select Lay Linear Algebra (the online textbook)
4. select a chapter
5. select a section
6. click study plan

Questions

1. Let A be an $n \times n$ matrix. Which of the following are equivalent to the statement that A is invertible?
   (a) rows of A span $\mathbb{R}^n$
   (b) rows of A are linearly independent
   (c) $A\vec{x} = \vec{b}$ has exactly one solution for all $\vec{b}$ in $\mathbb{R}^n$
   (d) $\det(A) \neq 0$, where $\det(A)$ is the volume of the papallelepiped formed by the columns of A
   (e) A^2 is invertible.

2. True or false. Explain your reasoning.
   (a) If a matrix is invertible, then it has an LU factorization.
   (b) If A^n is invertible for some given n, then A is also invertible.
   (c) If U is an echelon form of matrix A, then rank(U) = rank(A).
   (d) Any four linearly independent vectors in $\mathbb{R}^4$ forms a basis for $\mathbb{R}^4$.
   (e) The rank of an invertible $n \times n$ matrix is always n.
   (f) A square matrix is invertible if and only if the only solution to $A\vec{x} = \vec{0}$ is the trivial solution.

3. Suppose A is an $n \times n$ invertible matrix. Fill in the blanks.
   (a) The system $A\vec{x} = \vec{b}$ always has a ________________ solution.
   (b) The columns of A form a ________________ set.

(c) The columns of A ________________ $\mathbb{R}^n$.

(d) The only solution to $A\vec{x} = \vec{0}$ is ________________ .

(e) The transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$, is ________________ and ________________.

(f) The null space of A is ________________.

(g) The dimension of the column space of A is ________________.

4. For which scalars h, k does the matrix below fail to be invertible?

$$\begin{bmatrix} 0 & 1 & h \\ -3 & 10 & 0 \\ 3 & 5 & k \end{bmatrix}$$

5. For what values of k, if any, do the vectors not form a basis for $\mathbb{R}^4$?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ k \end{bmatrix}.$$

6. The determinant $\begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 5$. What are the determinants below?

(a) $\begin{vmatrix} 4 & 5 & 6 \\ a & b & c \\ 1 & 2 & 3 \end{vmatrix}$

(b) $\begin{vmatrix} 4 & 6 & 5 \\ a & c & b \\ 1 & 3 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 2a & 2b & 2c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

7. Calculate the LU decomposition of the matrix A, and use it to find the solution to $A\vec{x} = \vec{b}$, given below:

$$A = \begin{bmatrix} 1 & -3 & -1 \\ -1 & 5 & 4 \\ 3 & 1 & 11 \\ -3 & 1 & -11 \\ -2 & 0 & -2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 9 \\ -16 \\ -7 \\ 3 \\ -2 \end{bmatrix}$$

8. Use a determinant to determine whether the vectors are linearly independent.

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

9. Let A, B and C be 4×4 matrices with $\det A = 2, \det B = -3, \det C = 5$. Determine the value of the determinants of the following matrices.

$$AB, \quad AC^{-1}B, \quad B^TC^2, \quad A^3B^{-1}C^T, \quad 4C,$$

10. Matrix A and its LU decomposition are given below.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -3 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & 4 & -3 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Construct a basis for the null space of A. Do not compute A.

11. Construct a basis for the subspace

$$D = \{\vec{x} \in \mathbb{R}^5 \ : \ 2x_1 + 6x_2 + 12x_3 - 2x_5 = 0\}.$$

12. Suppose A is an 11×3 matrix and that $T(x) = Ax$. If T is one-to-one, the dimension of the null space of A is ________________.

13. True of False? Circle your answer on the left. Explain briefly.

True **False** The matrix $\begin{bmatrix} 0 & 0 & -3 & 1 & 2 \\ 10 & 5 & -4 & 0 & 8 \\ 0 & 0 & -3 & 3 & 6 \\ 10 & 5 & -3 & 1 & 7 \\ 2 & 1 & -2 & 1 & 6 \end{bmatrix}$ is invertible.

True **False** The matrix $\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & -\pi & 0 & 0 & 0 \\ -300 & 5 & -4 & 2 & 0 \\ 2 & 2016 & 0 & 1 & \pi^5 \end{bmatrix}$ is invertible.

True **False** The rank of a matrix A equals the number of pivot positions of A.

True **False** If all entries of an $n \times n$ matrix are greater than zero, then the determinant is greater than zero.

True **False** It is possible that a 3×6 matrix has a 4-dimensional nullspace.

True **False** The line with equation $y = 2x - 3$ is a subspace of $\mathbb{R}^2$.

True **False** If A and B are invertible $n \times n$ matrices, then $\det(ABA^{-1}) = \det(B)$.

True **False** If A has more columns than rows then the dimension of $Null(A)$ cannot be zero.

14. Consider the following matrix and its echelon form:

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 6 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of A?

(b) What is the dimension of the nullspace of A?

(c) Find a basis for $Col(A)$.

(d) Find a basis for $Null(A)$.

(e) Is A invertible?

15. Give an example of two matrices, A and B, such that $AB = 0$ but $A \neq 0$ and $B \neq 0$.

16. Consider the linear transformation.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_2 \\ x_1 + x_3 \\ x_2 - x_1 \end{bmatrix}$$

Complete the following formula for T^{-1}.

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \rule{4cm}{0.4pt} \\ \rule{4cm}{0.4pt} \\ \rule{4cm}{0.4pt} \end{bmatrix}$$

17. (a) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -7 \\ -3 \\ -11 \end{bmatrix}$. Note that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for a plane H in $\mathbb{R}^3$.

Decide if $\mathbf{v}$ is contained in H. It is, find the $\mathcal{B}$-coordinates of $\mathbf{v}$.

(b) What does the previous part tell you about whether or not

$$\begin{bmatrix} 1 & 0 & -4 \\ 3 & 1 & 1 \\ -7 & -3 & -11 \end{bmatrix}$$

is invertible? Explain briefly.

18. Compute the following determinant. Hint: using row reduction will greatly simplify the calculations.

$$\begin{vmatrix} -2 & -3 & -2 & 3 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 5 \\ -1 & -1 & -1 & 0 \end{vmatrix}$$

19. If possible, give an example of a 3×4 matrix A, in row reduced echelon form, with the given properties. If any are not possible, state why.

(a) rank(A) = 3

(b) rank(A) = 4

(c) dim(Null(A)) = 2

(d) dim(Null(A)) = 2, and rank(A) = 1