Exam 2 Spring '17

1. *Two parts.* If $T : \mathbb{R}^5 \to \mathbb{R}^4$ is a linear transformation with standard matrix A, and the dimension of the null space of A is $\dim \text{nul}(A) = 3$, then what is the dimension of the range of T? Justify your answer for full credit. Also, describe the range of T geometrically. (8 pts.)

2 pivot 5 3 free

$$4\left\{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}\right.$$

range of T = col(A)

dim col(A) = rank(A) = 2.

So dim range of $T = 2$

range of T is a plane in $\mathbb{R}^4$.

2. Find a basis for the null space of the matrix A. (12 pts.)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$A \sim \begin{array}{c} \begin{array}{ccccc} & r & & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x = -t$
$y = r$ (free)
$z = 0$
$u = s$ (free)
$v = t$

$$X = t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

basis

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

3. Find a basis for W. (8 pts.)

$$W = \text{span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}}, \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}} \right\}.$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -2 \\ 0 & -8 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

free

basis for W

$$\boxed{\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}}$$

4. For the three parts of this problem use the matrix A below.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ -4 & 6 & 1 \\ 10 & 3 & 7 \end{bmatrix}$$

(a) Find the LU decomposition of A. (10 pts.)

$$A \underset{\substack{2R_1+R_2 \\ -5R_1+R_3}}{\sim} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 12 & 1 \\ 0 & -12 & 7 \end{bmatrix} \underset{R_2+R_3}{\sim} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 12 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \underline{U}$$

$$\underline{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix}$$

(b) Find the determinant of A. Show your work. *Hint: use part (a)* (8 pts.)

$$\det(A) = \det(U) = 2 \cdot 12 \cdot 8 = \boxed{192}$$

(c) Find all solutions to $A\mathbf{x} = 0$. Justify your answer for full credit. (4 pts.)

$x = 0$ is the <u>only soln</u> since

A is invertible.

5. *True or False.* If v_1, v_2, v_3 are vectors in $\mathbb{R}^3$ such that none are scalar multiples of each other, then the set $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$. Either give a counterexample and explain why the statement is false, or give a clear justification for why the statement is true.

(8 pts.)

False. e.g.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ are lin dep so do not span $\mathbb{R}^3$, but none ~~of~~ the vectors in this set are scalar mult. of the others.

6. Find the inverse of A. Check your answer by matrix multiplication for full credit.

(14 pts.)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}$$

Check

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Find the standard coordinates of the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\}$. (8 pts.)

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix}_{\mathcal{B}} = 3\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 \\ 9+0 \\ -3-4 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 9 \\ -7 \end{bmatrix}}$$

8. Suppose the determinant of the matrix $\begin{vmatrix} a & b & c \\ 4 & 3 & 2 \\ 1 & 5 & 6 \end{vmatrix} = 5$. What is the determinant of

A ← $\begin{bmatrix} 4 & 3 & 1 \\ 1 & 5 & 6 \\ 3a & 3b & 3c \end{bmatrix}$? (8 pts.)

$$\begin{bmatrix} a & b & c \\ 4 & 3 & 2 \\ 1 & 5 & 6 \end{bmatrix} \sim^{3R_1} \begin{bmatrix} 3a & 3b & 3c \\ 4 & 3 & 2 \\ 1 & 5 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 3 & 1 \\ 1 & 5 & 6 \\ 3a & 3b & 3c \end{bmatrix}$$

So $\det(A) = 3 \cdot 5 \cdot (-1)$

$= \boxed{15}$

9. Suppose A is a 3×3 matrix and $\det(A^2) = 1$. Justify your answer to the following questions for full credit. (3 pts. each)

(a) *True* or *False*. A is invertible.

since $\det(A) = 0 \implies \det(A^2) = 0$

A must have $\det(A) \neq 0$

So A is invertible.

(b) *True* or *False.* The determinant of A is $\det(A) = 1$.

False. A could have $\det(A) = \pm 1$.

(c) *True* or *False.* The columns of A span $\mathbb{R}^3$.

Since A is invertible, the cols span $\mathbb{R}^3$ by I.M.T.

(d) *True* or *False.* There are infinitely many solutions to $A\mathbf{x} = \mathbf{b}$ for some choice of $\mathbf{b}$ in $\mathbb{R}^3$.

Since A is invertible, $Ax = b$ has exactly one soln $x = A^{-1}b$ for ALL b in $\mathbb{R}^3$.