Additional Problems for Midterm 3 Review

About This Review Set

As stated in the syllabus, a goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, solutions are not are provided for the additional review problem sets. This is intentional: upper level courses often don't have recitations, let alone worksheets and worksheet solutions. So students need, develop, and use various strategies to check their solutions in those courses. In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses.

The MML Study Plan

If you would like to prepare for the midterm by solving problems that have solutions, the MML Study Plan has hundreds of problems you can solve. MML will tell you if your work is correct and offers a few different study aids. To access problems for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your Math 1553 course
3. select Lay Linear Algebra (the online textbook)
4. select a chapter
5. select a section
6. click study plan

Questions

1. True or false. Explain your reasoning.
   (a) An eigenvector of a matrix could be associated with two distinct eigenvalues.
   (b) If A has eigenvalue λ, then A^T has eigenvalue λ.
   (c) The 3×3 zero matrix is diagonalizable.
   (d) Every non-zero vector in the null space of A is an eigenvector.
   (e) The only 2×2 matrix that only has eigenvalue zero is the 2×2 zero matrix.
   (f) If A is a diagonalizable $n \times n$ matrix, then $\text{rank}(A) = n$.
   (g) If 2 is an eigenvalue of A, and A is $n \times n$, then the rank of $A - 2I$ is $n - 1$.
   (h) The eigenvalues of a triangular matrix are its diagonal entries.
   (i) To calculate the eigenvalues of a square matrix A, we can row reduce A to triangluar form and then read off the eigenvalues.
   (j) A matrix A and its echelon form have the same eigenvalues.
   (k) A diagonalization of a matrix is unique.
   (l) If $\vec{v} \in \mathbb{R}^n$ is an eigenvector, then all of its elements cannot be equal to zero.
   (m) A real eigenvalue of a square matrix cannot be zero.
   (n) If A is $n \times n$, and the rank of $A + 3I$ is n, then -3 is an eigenvalue.
   (o) If 2 is an eigenvalue of A, and A is invertible, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

2. If possible, give an example of a matrix with the properties.

   (a) A 4×4 matrix whose entries are all real, but whose eigenvalues are complex.

   (b) A diagonalizable 3×3 matrix that has only two distinct eigenvalues.

   (c) A 2×2 non-zero matrix that has 0 as an eigenvalue twice.

   (d) A 2×2 matrix whose eigenvalues are 1, 4, and 0.

   (e) A 3×3 matrix whose eigenvalues are 1, 4, and 0.

3. The characteristic polynomial of a matrix A is $(\lambda - 3)(\lambda - 1)^2(\lambda + 4)^3$.

   (a) This means that det________________ $= (\lambda - 3)(\lambda - 1)^2(\lambda + 4)^3$.

   (b) The dimensions of A are ________________.

   (c) The eigenvalues of A are ________________.

   (d) Is A invertible?

4. Consider the matrices below.

$$X = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

   For both matrices, answer the following questions.

   (a) Determine all the eigenvalues.

   (b) Determine the dimension and basis for all eigenspaces.

   (c) Is the matrix diagonalizable? If so, construct a diagonalization.

5. A has only two distinct eigenvalues, which are 0 and 1. If possible, construct matrices P and D such that $A = PDP^{-1}$.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

6. Construct a diagonaliztion for $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$. Note that the matrix has complex eigenvalues.

7. Use the following equation to answer the questions below. *Do not do significant calculations. All questions can be answered with little or no work.*

$$A = \begin{bmatrix} -1 & 1 & 1 & -2 \\ 4 & 2 & -1 & -14 \\ 4 & -1 & 2 & 2 \\ 4 & 1 & 1 & -7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & -2 & -1 & 2 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & -2 & -1 & 2 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}^{-1}$$

   (a) Write down the characteristic polynomial of A.

   (b) List the eigenvalues of A with their multiplicities.

   (c) For each eigenvalue, find the *dimension* of the corresponding eigenspace. (You do *not* need to calculate the eigenvectors. Only the *dimensions* of the eigenspaces.)

   (d) Is A diagonalizable? Why or why not?

8. Suppose $\mathbf{v} = \begin{bmatrix} 1+3i \\ 2-i \end{bmatrix}$ is an eigenvector of a real 2×2 matrix A. Provide another eigenvector of A that is linearly independent of $\mathbf{v}$.

9. (a) Let B be a 2×2 matrix with two identical rows. Explain why zero is an eigenvalue of B.
   (b) Suppose again that B is a 2×2 matrix with two identical rows. Find a relationship between the two entries in each row such that the second eigenvalue of B is 1.

10. Let
$$A = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}.$$
   (a) Find the eigenvalues of A.
   (b) Find the eigenvectors of A.
   (c) Write A as a product two matrices so that one of them describes a scaling and the other a rotation.
   (d) By how much does A scale?
   (e) By how much does A rotate?
   (f) Find A^{2016} without diagonalizing A. (Hint: use the fact that you know how much A scales and rotates.)

11. Let
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
   (a) Find the eigenvalues of A. What is the algebraic multiplicity of each eigenvalue?
   (b) Determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
   (c) Compute A^{114} and A^{115}.

12. Consider the dynamical system
$$\vec{x}_k = A\vec{x}_{k-1} = \begin{pmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{pmatrix} \vec{x}_{k-1}, \quad k = 1, 2, 3, \dots$$
The eigenvalues of A are 1 and $\frac{1}{4}$. Analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.