Week 3. Monday

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Today

* Span
* linear combination
* geometric intepretation
* $Ax=b$ vs. $Ax=0$

and what are those

The matrix equation $\boxed{Ax=b}$

means the same thing as $[A \mid b]$

row 1 col 3

means the same thing as

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

row 1 col 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

row

a_{ij}

column

Ex.

$$\underbrace{\begin{aligned} 3x - 2y &= 4 \\ 4x + y &= 8 \end{aligned}}_{\text{in eqn form}}$$

vs. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 4 & 1 & 8 \end{array}\right]$ } augmented matrix

vs. $\underbrace{\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}}_{\text{Matrix eqn.}}$.

Most Important Formula for the near future

$$A\,X \overset{\text{defn}}{=} V_1 \cdot x_1 + V_2 \cdot x_2 + \dots + V_n \cdot x_n$$

where $A = [V_1 \; V_2 \; V_3 \cdots V_n]$ $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Ex. Write out

$$\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} + y \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

this is an example of a linear combination of the vectors $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Defn. A linear combination of the vectors $V_1, V_2, \dots, V_n$ is any vector of the form

$$c_1 V_1 + c_2 V_2 + \dots + c_n V_n \quad \text{where } c_i \in \mathbb{R}.$$

Ex. $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is a lin. comb. of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ & $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Q: Is $\begin{bmatrix}1\\-2\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}2\\1\end{bmatrix}$?

Looking for $x, y \in \mathbb{R}$

$$\underbrace{x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}}$$

guess & check? $x = y = 1$

$$\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} \neq \begin{bmatrix}1\\-2\end{bmatrix}$$

So we need look harder for x, y that will work.

write as

$$Ax = b$$

$$\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$$

Or write as

$$\left[\begin{array}{cc|c}1 & 2 & 1\\0 & 1 & -2\end{array}\right]$$

$$\sim_{-2R_1+R_2} \left[\begin{array}{cc|c}1 & 0 & 5\\0 & 1 & -2\end{array}\right]$$

Check $x = 5, y = -2$

$$5\begin{bmatrix}1\\0\end{bmatrix} + (-2)\begin{bmatrix}2\\1\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}1\\-2\end{bmatrix} \checkmark$$

Sanity check.

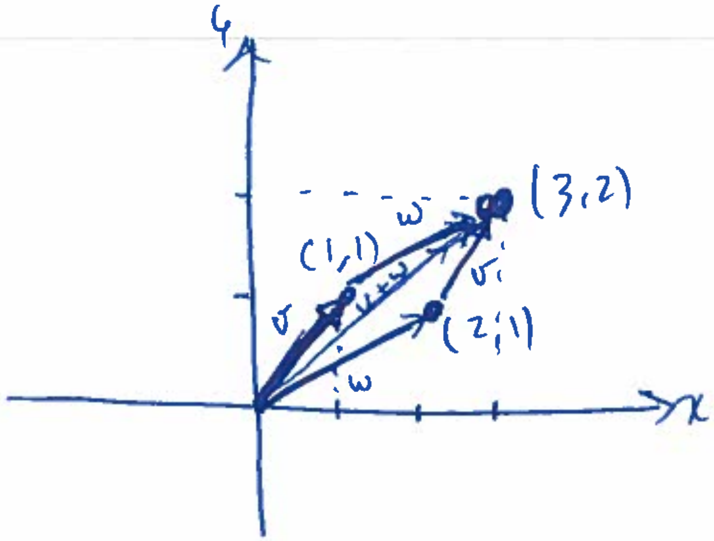
$$x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$$

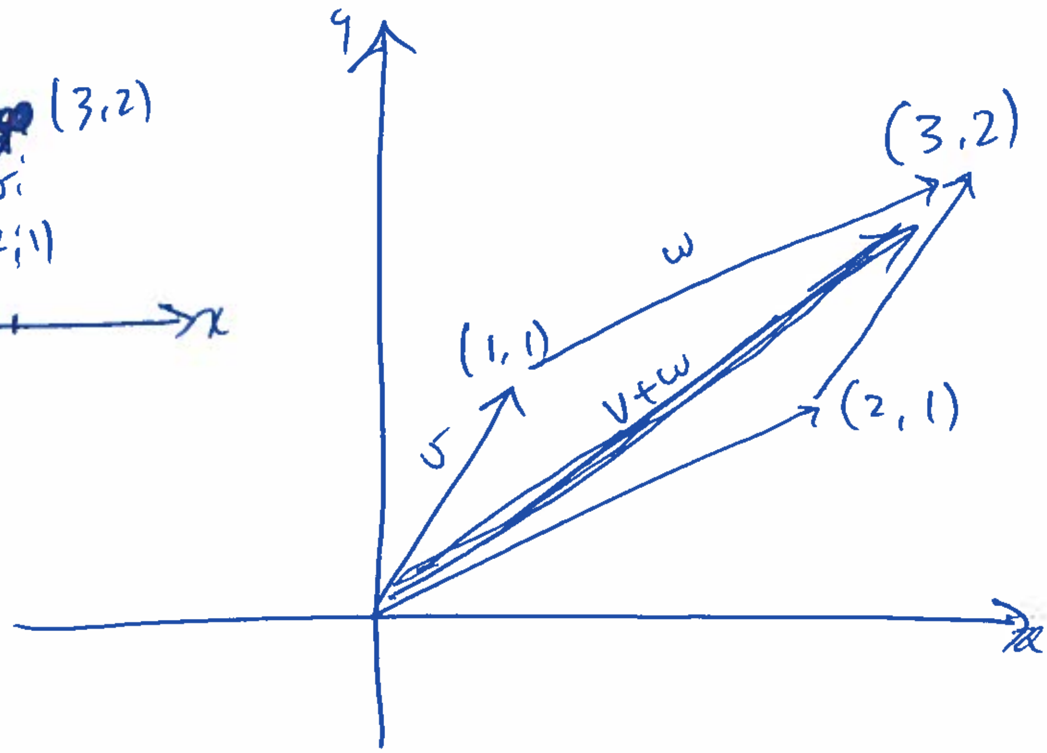
$$\Leftrightarrow \begin{bmatrix}x\\0\end{bmatrix} + \begin{bmatrix}2y\\y\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x+2y\\0+y\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix} \Leftrightarrow \begin{cases}x+2y=1\\ y=-2\end{cases}$$

What does $\underset{v}{\begin{bmatrix}1\\1\end{bmatrix}} + \underset{w}{\begin{bmatrix}2\\1\end{bmatrix}}$ look like?

$$\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

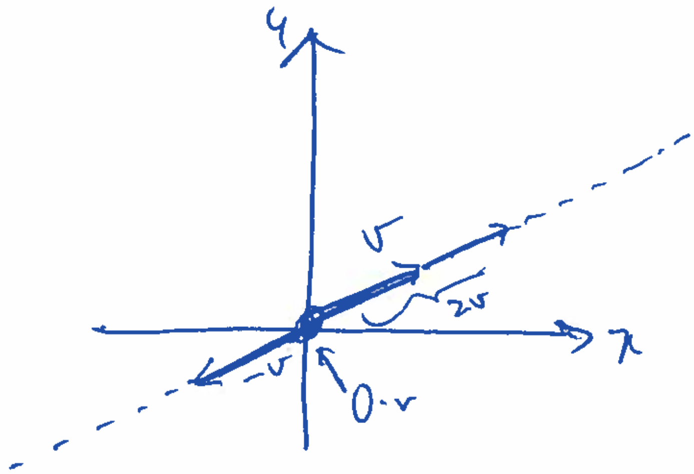




$v+w$
$v-w$
v
w
$2v$
$2v-3w$

These are all various linear combinations of v & w

Q_2: What is the collection of all vectors that are a linear combination of v & w?



Q_1: What are all the vectors you can get as a linear combination of v?

The <u>span</u> of v is a line.

<u>Defn.</u> The span$\{v_1, v_2, \ldots, v_n\}$ is the set of <u>ALL</u> linear combinations

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \quad , \quad c_i \in \mathbb{R}.$$

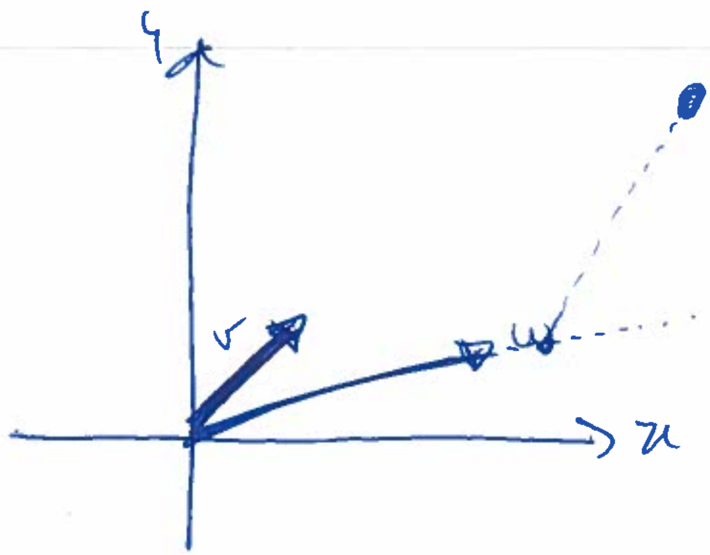
"belongs to"

Q: $\boxed{\text{Is } \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} ?}$

Is the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the span of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

Q: Are there any vectors that are **NOT** in $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$?

$$\underset{v}{\uparrow} \qquad \underset{w}{\uparrow}$$



No. All $b \in \mathbb{R}^2$ are in the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 1 & 1 & b \\ 1 & 2 & a \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & a \\ 1 & 1 & b \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -1 & b-a \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & -b+a \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & a-2(-b+a) \\ 0 & 1 & -b+a \end{array}\right]$$

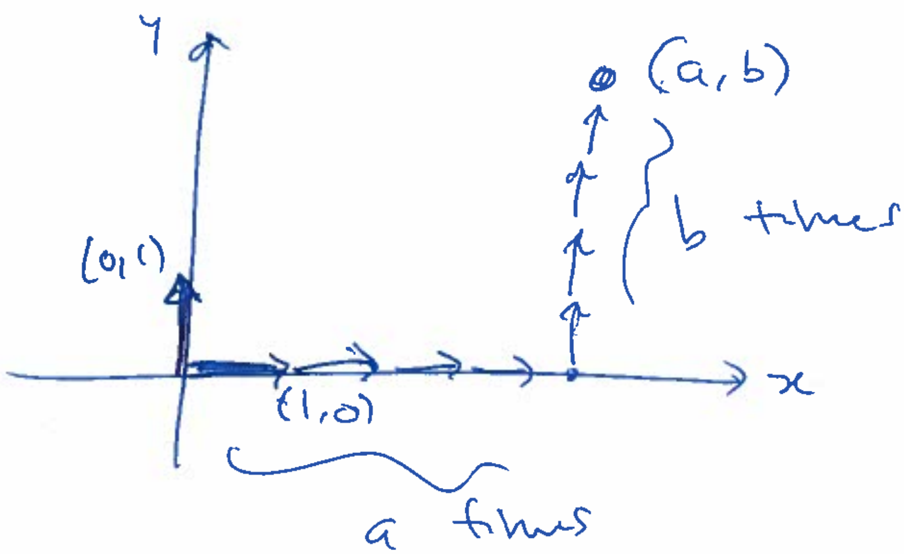
$$= \left[\begin{array}{cc|c} 1 & 0 & -a+2b \\ 0 & 1 & -b+a \end{array}\right].$$

$$(-a+2b)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-b+a)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} b \\ a \end{pmatrix}.$$

Q: Describe Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Is every $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ in the span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad ?$$



$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$$

because 1) Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ consists only of $\mathbb{R}^2$ vectors

2) every vector in $\mathbb{R}^2$ is in the span.

In General

$$\text{Span}\{v_1, \dots, v_n\} = \mathbb{R}^n \quad \text{if}$$

$A = [v_1 \; v_2 \dots v_n]$ has a pivot in every row

Q: I want v, w, b such that

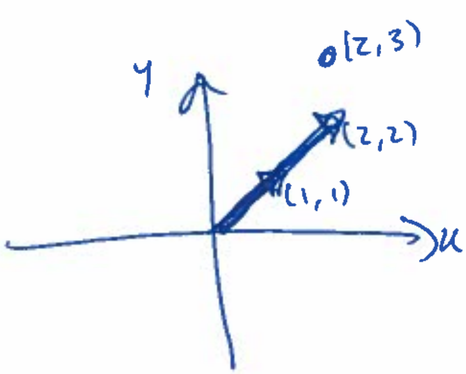
$$b \notin \text{Span}\{v, w\}$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \quad (\checkmark)$$

Does it happen that

$$x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ for some } x, y?$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & -1 \end{array}\right]$$



$$\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is not on this line.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are called "linearly dependent."

Wednesday

Today:

* check understanding of
  span, linear combination, $Ax=b$, $Ax = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$
* new material: Linear Independence

Ex. Solve the matrix equation

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Step 1: write it as an augmented matrix & solve (row reduce)

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 3 & 7 \end{array}\right]$$

$$\sim_{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 3 & 3 \end{array}\right]$$

$$\sim_{\frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Step 2: Express your answer in a similar form as to how the question was stated

Ans: $\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

we just found that

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

using the eqn $Ax = x_1 v_1 + x_2 v_2$

$$(2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 7 \\ 2 \end{bmatrix} \checkmark$$

(true eqn)

Also, this says

① " $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ "

It verifies the statement

② " $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ is in the span of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ "

b/c

① linear comb. of v_1, v_2 is any vector of the form

$$c_1 v_1 + c_2 v_2 \quad , \quad c_i \in \mathbb{R}$$

② The span of v_1, v_2, written

$$\text{span}\{v_1, v_2\} = \{c_1 v_1 + c_2 v_2 : c_1, c_2 \in \mathbb{R}\}$$

is the set of all linear combinations of v_1, v_2.

Ex. What about $\mathbb{R}^3$?

Is $\begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$ in the span of $\left[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right]$?

Soln.

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 7 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 3 & 0 & 7 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

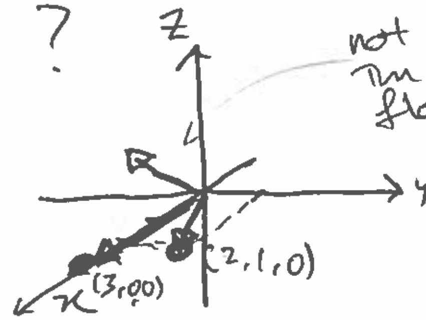
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

$$2 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \left(\tfrac{1}{2}\right) \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}.$$

So $\boxed{\text{yes.}}$

Ex. Is $\begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$?

not in the floor!



Soln. (easy if you see "it")

Defn: The vectors $v_1, v_2, \ldots, v_k$ are said to be linearly independent if the only way to have

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

is for all of $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

Ex. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ are these linearly independent? (or are they linearly dependent)

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 0$$

Are there solutions other than the "obvious" one.

The trivial solution $c_1 = c_2 = 0$ certainly work, but are there other solutions?

Q: How many solutions does the system

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0} \quad \text{have?}$$

The "zero vector"

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

only soln is $c_1 = c_2 = 0$

Ex.

$$\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}7\\2\end{bmatrix}$$ are they lin ind?

Soln. we saw that $\begin{bmatrix}7\\2\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$

in particular

$$2\cdot\begin{bmatrix}2\\1\end{bmatrix} + 1\cdot\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}7\\2\end{bmatrix}.$$

So

$$2\begin{bmatrix}2\\1\end{bmatrix} + 1\cdot\begin{bmatrix}3\\0\end{bmatrix} - 1\cdot\begin{bmatrix}7\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Decide for lin ind. if $Ax=0$ has a non-trivial soln.

Q: (conceptual)

Can you give me an example of 3 vectors in $\mathbb{R}^2$ which linearly independent?

Soln.

$$Ax=0: \left[\begin{array}{ccc|c} * & * & * & 0 \\ * & * & * & 0 \end{array}\right] \sim$$

Q: (conceptual)

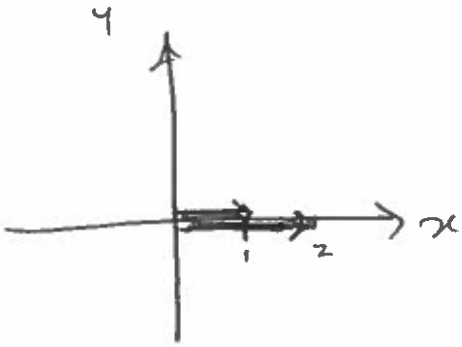
What's the relationship between $b \in \text{Span}\{v, w\}$ and the set $\{v, w, b\}$ being linearly ind/dep?

Soln. If $b \in \text{Span}\{v, w\}$ then $b = c_1 v + c_2 w$ for some c_1, c_2

So $c_1 \cdot v + c_2 \cdot w - b = 0$, and $\{v, w, b\}$ is linearly dependent

Q: What about

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are they lin ind?



No. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

is a recipe showing a non-~~trivial~~trivial linear comb to get 0.

Q: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are they lin ind?

$Ax=0$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$$c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Is this possible without $c_1 = c_2 = c_3 = 0$?

Sure. e.g. $c_1 = 0$, $c_2 = 0$ and $c_3 = 1$ (anything).

So $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ should NEVER be in a linearly ind. set!