Today: Week 4 1/30

* linear idependence
* basis

Defn: (Recall) A set of vectors $\{v_1, v_2, \dots, v_n\}$ is linearly independent if the system

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \quad \left\{ \begin{array}{l} \text{AKA.} \\ \Leftrightarrow [v_1\ v_2 \dots v_n \mid 0] \\ \Leftrightarrow [v_1\ v_2 \dots v_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \right\}.$$

has only the trivial solution $c_1 = c_2 = \dots = c_n = 0$.

For example, none of the vectors should be in the span of the others.

Ex. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are linearly dependant

b/c $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \text{span} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (& visa versa)

Ex. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ are linearly ind. themselves

so $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ are lin. independent.

New defn: **BASIS** is the single most defn of the whole semester (by far)

Defn. A basis for a subspace W in $\mathbb{R}^n$ is a set of vectors $\{v_1, v_2, \dots, v_k\}$ such that

① $\text{Span}\{v_1, v_2, \dots, v_k\} = W$

(in English: The span of the vector is the subspace)

② The vectors $\{v_1, \dots, v_k\}$ are linearly independent.

Ex. Verify that

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ are NOT a basis for $\mathbb{R}^2$.

First note $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is a linearly dependent set of vectors, so ② fails.

So $\mathcal{B}$ is NOT a basis since it is not a linearly independent set.

Additionally (extra) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ does NOT span $\mathbb{R}^2$ either!

How to check?

e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \notin \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ $\quad \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 1 \end{array}\right]$

<u>Ex.</u> The set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

This is actually

<u>THE</u> Standard basis of $\mathbb{R}^2$

$$\boxed{\begin{aligned} e_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ e_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}}$$

<u>Check.</u> ($\mathcal{B}$ spans $\mathbb{R}^2$)

Pick arbitrary $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$: $\underline{a \in \mathbb{R}, b \in \mathbb{R}}$

any real #'s for a, b.

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Sure set

$c_1 = a$

$c_2 = b$ (easy!)

<u>Check</u> ($\mathcal{B}$ is lin ind)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ already rref

pivot in every ~~row~~ column.

$\Longleftrightarrow$ vectors are lin ind.

<u>FACT:</u> $\{v_1, v_2, \ldots, v_n\}$ are lin. ind

$\Longleftrightarrow$ $A = [v_1 \; v_2 \ldots v_n]$ has a pivot in every column

Ex. Is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Is this a basis of $\mathbb{R}^3$?

Check (linear ind.)

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

how many solns?

Is the only soln the vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{array}{ccc} \text{pivot} & \text{pivot} & \text{free } t \\ \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

two pivot columns
one free column

$Ax = 0$ solns

$x = -t$
$y = -2t$
$z = t$ (free)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Set $t = 1$.

$c_1 = -1 \quad c_2 = -2 \quad c_3 = 1$

$$\underbrace{-1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{non-trivial linear combination}} \overset{\checkmark}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↑ the zero vector.

Ex. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ What is this a basis for?

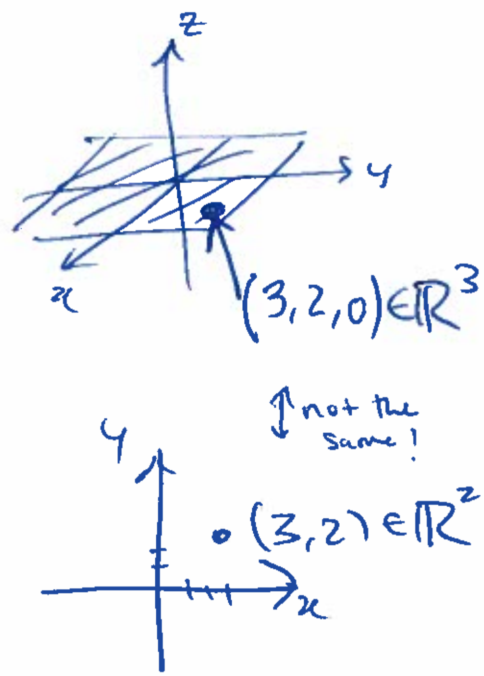
It is certainly a set of linearly ind. vectors.

So it is a basis for

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = W$$

all the vectors in the span look like $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, zero z-comp.

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$



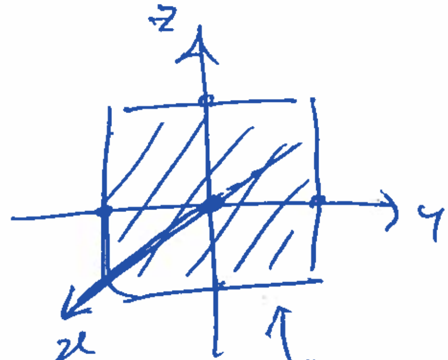
Q: Is W the same thing as $\mathbb{R}^2$?

A: No. It's like $\mathbb{R}^2$ in some way which we make precise when we talk about "isomorphisms"

one-to-one & onto linear maps

<u>Ex</u>. Q_1: Is $\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ basis for $\mathbb{R}^2$?

Q_2: Find a basis for the z-y-plane in $\mathbb{R}^3$



Ans 1: No. They <u>span</u> $\mathbb{R}^2$ but are not lin ind.

Ans. 2: $\left\{ \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

$\uparrow e_2 \qquad \uparrow e_3$

z-y-plane satisfies <u>$x = 0$</u>.

<u>Defn</u>: The dimension of a subspace W is the number of vectors in <u>ANY</u> basis of W.

<u>Ex</u>. The dimension of $\mathbb{R}^2$ is 2
The dim. of $\mathbb{R}^3$ is 3

<u>why</u>
$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$
$\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$
are bases w/ 2 & 3 vectors resp.

The dim of "the floor" $z=0$ in $\mathbb{R}^3$ is equal to 2

b/c $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ are a basis for the floor.

dim $\mathbb{R}^n = n$ too, th...

FACTS:

① If you have m vectors then the span of them will have dim $\leq m$.

② The span of m vectors will have dimension equal to m if they are lin. ind.

Conclusions

① The number basis vectors in $\mathbb{R}^n$ is always equal to n.

e.g. $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$ is NEVER A BASIS For a 2-dimensional Subspace (like $\mathbb{R}^2$).

e.g. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ is NEVER a BASIS for a 3-dim space (like $\mathbb{R}^3$)

Defn. A <u>subspace</u> W of $\mathbb{R}^n$ is a collection of vectors must satisfy two conditions.

1) $v, w \in W \Rightarrow v + w \in W$ (vector addition)

2) $v \in W,\ c \in \mathbb{R} \Rightarrow cv \in W$ (scalar mult).

Today

Week 4 Wed 2/1

NOTE: Basis & subspace were PREVIEW material meant to help make sense of lin. comb./span.
Basis & subspace are NOT covered on Exam 1.

- linear transformations } §1.8 (MyLab due tom.)
- domain, range, codomain }
- Standard matrix of lin. trans.} §1.9 (MyLab due next week)

EXAM 1 in one week (from Friday)

Covers CHAPTER 1

- Solving systems of lin. eqns
- One unique soln, inf. many, no soln
- consistent vs. inconsist
- parametric eqn form (free variables)
- parametric vector form (similar w/ vectors)
- linear combination, span, linear independence
- rref, ref, row operations
- linear transformations (today, Friday, next week)

Defn: A linear transformation ~~is~~ $T(x)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ which satisfies two conditions

1) $T(v+w) = T(v) + T(w)$
2) $T(c \cdot v) = c \cdot T(v)$.

Ex. $T: \mathbb{R} \to \mathbb{R}$

options: are they linear transformations or not.

$T(x) = x$ (identity)

yes b/c $T(v+w) = v+w = T(v) + T(w)$
& $T(cv) = c \cdot v = c \cdot T(v)$.

$T(x) = x^2$

$T(1+2) = T(3) = 3^2 = 9$
$T(1) + T(2) = 1^2 + 2^2 = 1 + 4 = 5$

$5 \neq 9$ so breaks #1.

btw also breaks #2
e.g. $T(2 \cdot x) = (2x)^2 \neq 2 \cdot x^2 = 2 \cdot T(x)$

Ex. $T(x) = 3x+1$

Is this linear? (let's be more precise)
Is this a linear transformation?

<u>Ans.</u> No!

$$T(1+2) = \cancel{(3(x)+1)+1}$$
$$T(3) = 3(3)+1 = \underline{10}$$
$$T(1)+T(2) = (3(1)+1) + (3(2)+1)$$
$$= 4+7 = \underline{\underline{11}}$$

<u>Ex.</u> Define

$$T(x) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This defines a function

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$ ← which is in $\mathbb{R}^3$

$\mathbb{R}^2$ is the domain (inputs)

$\mathbb{R}^3$ is the codomain (where outputs "live")

Ex. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$

Q: What is the range of T?

Defn. For a function $T : \mathbb{R}^m \to \mathbb{R}^n$

the domain is $\mathbb{R}^m$

the codomain is $\mathbb{R}^n$

and the range is all vectors which can be obtained as an output $b = T(x) \in \mathbb{R}^n$ for some (good choice) of $x \in \mathbb{R}^m$

Q: Give me ONE vector in the range of T.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For example

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (2)\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

So $\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ is in the range of T. (it's a potential output)

The other outputs $\underline{\underline{\text{all}}}$ take the form

notice

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$$

So the vectors in the <u>range</u> of T are <u>exactly</u> the vectors in the "column space" of T, which in this case ~~are~~ is the span of the columns of the matrix which defined T.

Q: Is $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ in the range of T, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \; ?$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

inconsistent system

equivalent question is

Is $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$?

Looking for x, y which satisfy

$$x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

(is this consistent?)

Ex.

Define

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y-z \\ y+z \end{bmatrix}$$

e.g.

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1+2-1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+2-0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Q: What is the domain? codomain? range?

$\mathbb{R}^3$ (the inputs come from $\mathbb{R}^3$)

$\mathbb{R}^2$ (the outputs are all in $\mathbb{R}^2$)

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y-z \\ y+z \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 2y \\ y \end{bmatrix} + \begin{bmatrix} -z \\ z \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

the range is $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$

7.

Ex. Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

Q1: Domain of $T(x) = Ax$?
~~Range~~ Codomain of $T(x) = Ax$?

Q2: Range of $T(x) \overset{?}{=} Ax$ (write as a span)

Q3: Is b in the range of T?
If so find x s.t. $T(x) = b$

Q4: Find the image of u under T.
(please compute $T(u)$)

A1: Domain is $\mathbb{R}^2$ since codomain is $\mathbb{R}^3$

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} X = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

must be $X = \begin{bmatrix} x \\ y \end{bmatrix}$

to make $Ax = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$ make sense

A2: range is $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} \right\}$

A4: yes

$$\frac{3}{2}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \left(\frac{-1}{2}\right)\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

So. $T\left(\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

A4: $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$