Today

Week 5 Mon 2/6

- linear transformations (geometrically & review)
- Standard matrix of a lin. trans.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} a \\ d \\ h \end{bmatrix} + y\begin{bmatrix} b \\ e \\ i \end{bmatrix} + z\begin{bmatrix} c \\ f \\ j \end{bmatrix}$$

Recall

$$Ax = x_1v_1 + x_2v_2 + \dots + x_nv_n$$

where $A = [v_1\ v_2 \cdots v_n]$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

So. e_1 = first standard basis vector.

first col.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \overset{\downarrow}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = 1\begin{bmatrix} a \\ d \\ g \end{bmatrix} + 0\begin{bmatrix} b \\ e \\ h \end{bmatrix} + 0\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix}$$

Similarly

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ e \\ h \end{bmatrix} \leftarrow \text{2nd col.}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ f \\ i \end{bmatrix} \leftarrow \text{3rd col.}$$

Q: Suppose $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - 3y + z \\ 4x - z \\ y + z \end{bmatrix}$

The standard matrix of T.

What is a matrix which "does the job"

$T(x) = Ax$? ANS. $A = [T(e_1)\ T(e_2)\ T(e_3)]$

$$T(e_1) = T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2-0+0\\4-0\\0+0\end{bmatrix} = \begin{bmatrix}2\\4\\0\end{bmatrix}$$

So the first col of A is $\begin{bmatrix}2\\4\\0\end{bmatrix}$

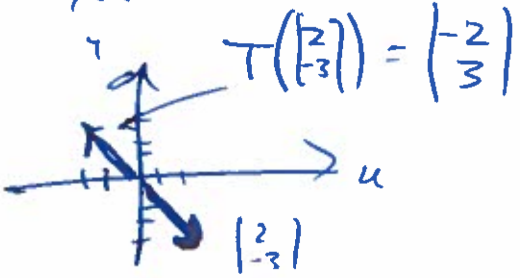
$$T(e_2) = \begin{bmatrix}-3\\0\\1\end{bmatrix}, \quad T(e_3) = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

So $A = \begin{bmatrix}2 & -3 & 1\\4 & 0 & -1\\0 & 1 & 1\end{bmatrix}$. The standard matrix of T.

Q: Define

$T: \mathbb{R}^2 \to \mathbb{R}^2$ as follows.

$T(x)$ = the vector with the same magnitude as x but on the opposite direction as x.



$T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right) = \begin{bmatrix}-2\\3\end{bmatrix}$

$$T(x) = -x$$

e.g. $T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right) = \begin{bmatrix}-2\\3\end{bmatrix}$

What is the standard matrix of T?

$A = \begin{bmatrix}\cdot & \cdot\\ \cdot & \cdot\end{bmatrix}$ it has to be 2×2 b/c

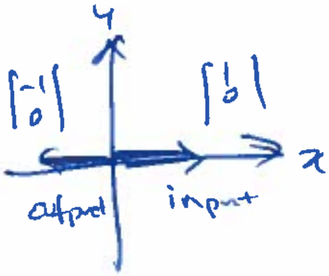
$\begin{bmatrix} | & | \end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}\cdot\end{bmatrix} + y\begin{bmatrix}\cdot\end{bmatrix}$

3

$$A = \begin{array}{cc} T(e_1) & T(e_2) \\ \downarrow & \downarrow \end{array}$$
$$A = \begin{bmatrix} ? & ? \end{bmatrix}$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$ b/c

$T(e_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

So
$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Sanity check.

$T(x) = -x$

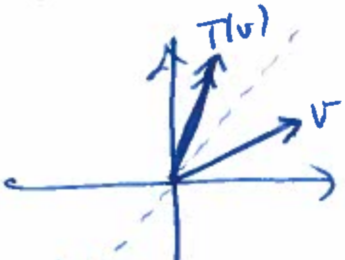
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} -x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

Ex. Find the standard matrix for the following linear transformations

1) $T: \mathbb{R}^2 \to \mathbb{R}^2$ T should scale each vector in $\mathbb{R}^2$ by a factor of 2.

ans: $\begin{array}{cc} T(e_1) & \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ & T(e_2) \end{array}$

2) $T: \mathbb{R}^2 \to \mathbb{R}^2$ T reflects vectors across the line $y = x$

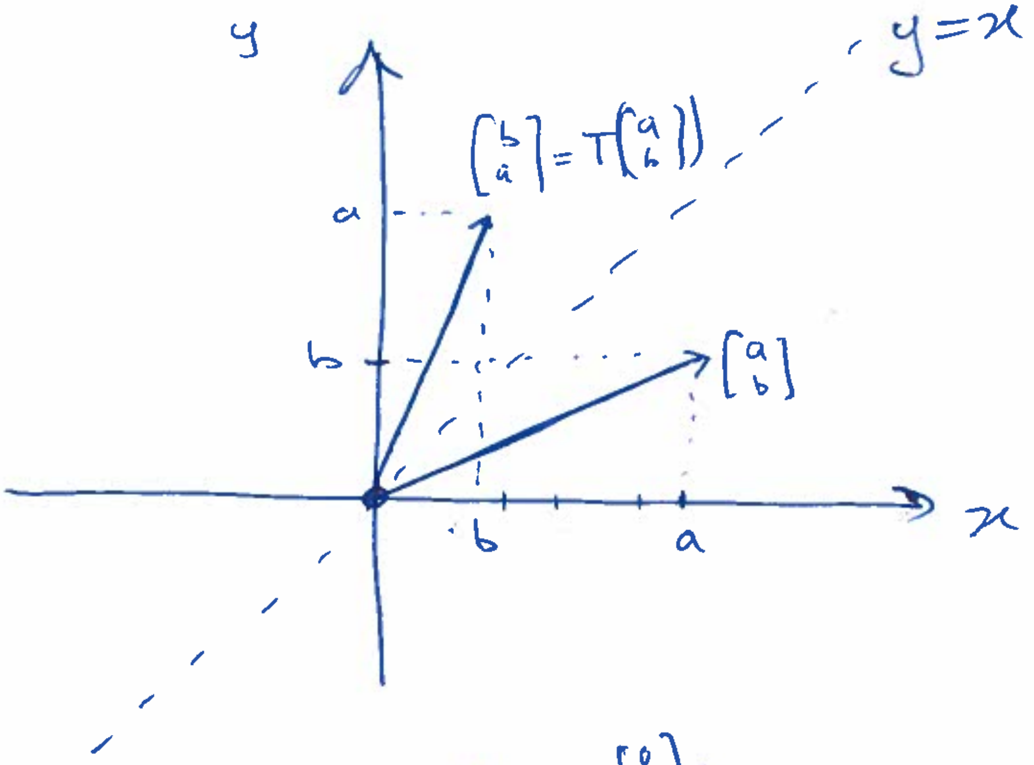


ans. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

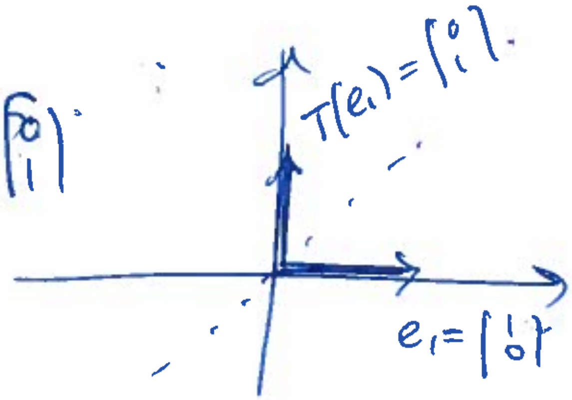
3) $T: \mathbb{R}^2 \to \mathbb{R}^2$ T rotates vectors in $\mathbb{R}^2$ by 90° counter-clockwise

ans $\begin{bmatrix} 0 & -1 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ reflect about line $y=x$.



$T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



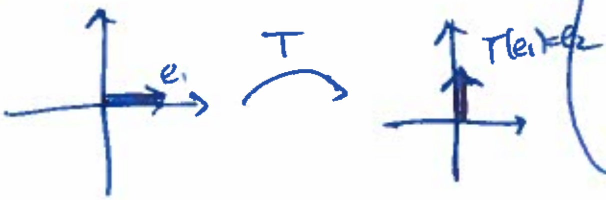
$T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Similarly.

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

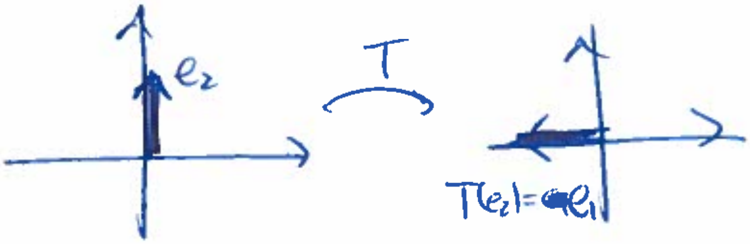
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

#3.

$T(e_1) = e_2$



$T(e_2) =$



$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

New Q:

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x\\y\\0\end{bmatrix}$$

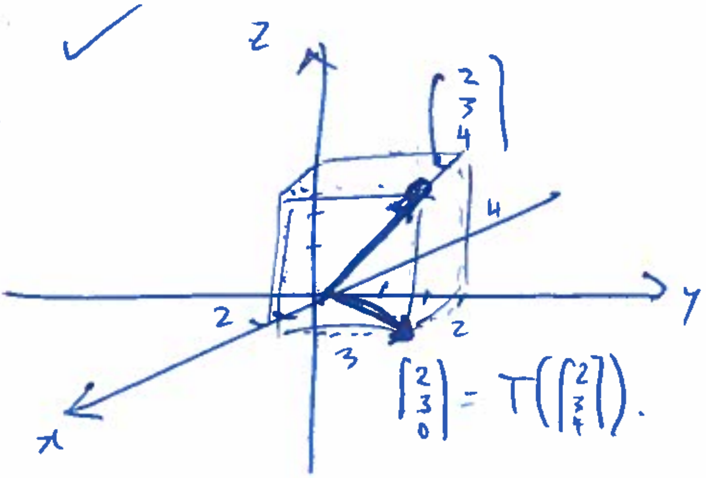
Find the standard matrix, give T a name, and state domain & codomain of T.

$$A=\begin{bmatrix}T(e_1) & T(e_2) & T(e_3)\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}.$$

Lets see

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$=\begin{bmatrix}x\\y\\0\end{bmatrix} \checkmark$$



$$\begin{bmatrix}2\\3\\0\end{bmatrix}=T\left(\begin{bmatrix}2\\3\\4\end{bmatrix}\right).$$

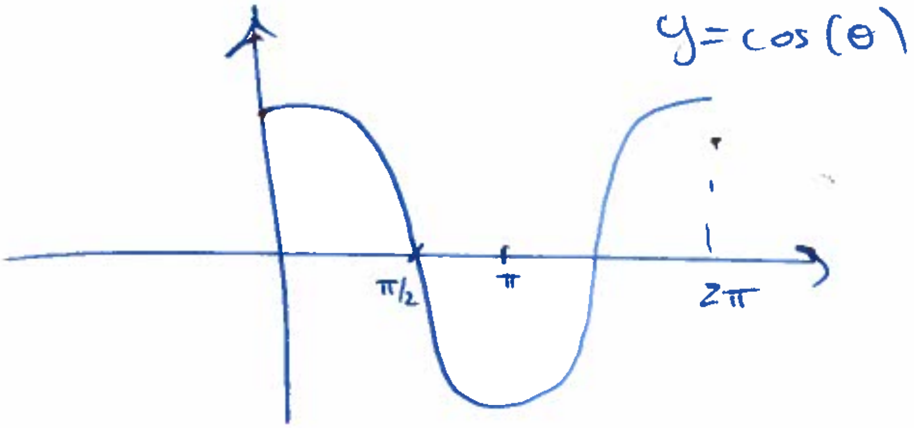
$T:\mathbb{R}^3\to\mathbb{R}^3$

T projects $\mathbb{R}^3$ to the x-y-plane

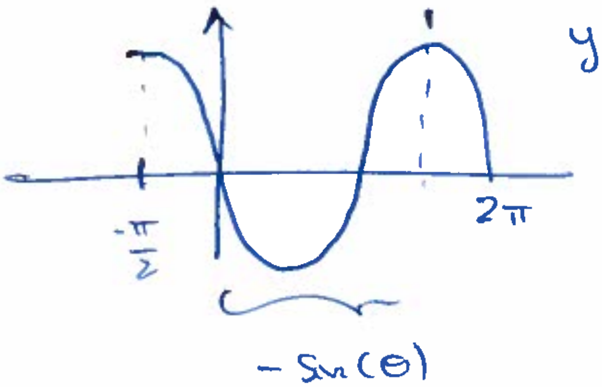
7.

Shift left

$$\cos(\theta + \pi/2) = -\sin(\theta)$$



Shift ~~[illegible]~~ left by π/2



$$y = \cos(\theta + \tfrac{\pi}{2}) = -\sin(\theta)$$

$$\sin(\theta + \pi/2) = \cos(\theta)$$