Week 6 Wed 2/15

Today:

* matrix mult
* weirdness of matrix mult.
* inverses
* transpose

New thing:

$$\underset{2\times 2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}}$$

e.g.

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}} = \begin{bmatrix} (1)(-1)+(0)(-1) & (1)(2)+(0)(3) \\ (1)(-1)+(2)(-1) & (1)(2)+(2)(3) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ -3 & 8 \end{bmatrix}$$

$$\overset{B}{\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} -1+2 & 0+4 \\ -1+3 & 0+6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}.$$

FACT In general matrix mult. is not commutative. So

$$AB \neq BA \text{ in general!}$$

(but sometimes it is true)

Ex. Set $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

compute $I \cdot A$ where $A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

and also $A \cdot I$.

$$\begin{bmatrix} \boxed{1 \quad 0} \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} -1 \\ 1 \end{matrix}} & \begin{matrix} 2 \\ 3 \end{matrix} \end{bmatrix} \overset{?}{=} \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{?}{=} \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}.$$

FACT: $I \cdot A = A \cdot I = A$.

For ANY matrix A.

FACT: If $A \cdot B = A \cdot C$

does it follow that $B = C$?

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \overset{?}{=} \overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \cdot \overset{C}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \;?$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

So $A \cdot B = A \cdot C$

but $B \neq C$.

New Thing: Transpose

Given an $m \times n$ matrix A, the matrix A^T called A transpose is the matrix where you ~~replace~~ exchange the rows w/ columns.

1rst row — now is the 1rst column.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

FACTS (you check them yourself) e.g. $A+B$

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$$

1) $(A^T)^T = A$

2) $(A+B)^T = A^T + B^T$

3) $(c \cdot A)^T = c \cdot A^T$

☆ 4) $(A \cdot B)^T \neq A^T \cdot B^T$

$(A \cdot B)^T = B^T \cdot A^T$

same

$$\underset{m \times n}{A} \cdot \underset{n \times l}{B} = \underset{m \times l}{C}$$

For general matrix mult.

$$A \cdot B = A \cdot [v_1\ v_2 \cdots v_n]$$

$$= [Av_1\ Av_2 \cdots Av_n]$$

columns of $A \cdot B$ are found by $A \cdot$(ith col of B)

Q1: is this product defined?

EX. Q2: what is the size of resulting matrix?

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 14 & 12 \\ 4 & 6 \end{bmatrix}}$$

A1: yes b/c left matrix has same # of columns as the right matrix has # of rows.

A2: 2x2 b/c it has the same number of rows as the left matrix, and same # of cols as the right matrix.

also can do this way

$$= \begin{bmatrix} 14 & 12 \\ 4 & 6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}}_{Av_1} \qquad \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}}_{Av_2}$$

Last weird fact:

If $A \cdot B = 0$

it may be possible that neither A nor B is the zero matrix.

e.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Inverses:

For numbers we have 1 (mult. identity).

$$a \cdot 1 = a \quad \text{for all } a \in \mathbb{R}.$$

Also, every non-zero $a \in \mathbb{R}$ has a "inverse" b so that

$$a \cdot b = 1.$$

(in particular $b = \frac{1}{a}$)

For matrices something similar happens (not the same)

If $A \cdot A^{-1} = A^{-1} \cdot A = I$

FACT: Then A^{-1} is called the inverse of A.

1) only square matrices can have inverses.

2) only ~~square~~ matrices with lin ind cols ~~don't~~ do ~~have~~ have an inverse.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

It has lin ind cols

So it has an inverse.

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

B has an inverse

C does NOT.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{Then} \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \frac{1}{1-0}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

$$B = \frac{1}{8-3}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

$$C \overset{?}{=} \frac{1}{2-2}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{0}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} ??$$

C doesn't have an inverse

Check

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \left(\frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Facts about inverses:

1) $(A^{-1})^{-1} = A$

2) $(A\cdot B)^{-1} = B^{-1}\cdot A^{-1}$

3) $(A^{T})^{-1} = (A^{-1})^{T}$

notice

$$(A\cdot B)(A\cdot B)^{-1} = I.$$

same.

also

$$A\cdot \underbrace{B (B^{-1}}_{I}\cdot A^{-1}) = A\cdot I\cdot A^{-1}$$

$$= A\cdot A^{-1}$$

$$= I$$

Algorithm for finding inverse of a 3x3.

$$[A \mid I] \sim \dots \sim [I \mid A^{-1}].$$

e.g. Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

Set up.

Find rref

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

A^{-1}