Week 7
today 2/20

From last week (wrap up)

Ex. Use INVERSES to solve the system

$$\begin{aligned} 7x + 11y &= 4 \\ 2x + 3y &= 5 \end{aligned} \quad \longleftrightarrow \quad Ax = b$$

row operations are gross.

$$[A|b] = \left[\begin{array}{cc|c} 7 & +11 & 4 \\ 2 & 3 & 5 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{array}\right]$$

better way? use inverses.

If we want to know what x is,

$$Ax = b \implies A^{-1} \cdot Ax = A^{-1} \cdot b$$

$$\implies (I) \cdot x = A^{-1} b$$

$$\implies x = A^{-1} b$$

Step 1: Find A^{-1}

Step 2: compute $x = A^{-1} b$.

$$A = \begin{bmatrix} 7 & +11 \\ 2 & 3 \end{bmatrix} \quad \text{so} \quad A^{-1} = \frac{1}{7 \cdot 3 - 2 \cdot 11} \begin{bmatrix} 3 & -11 \\ -2 & 7 \end{bmatrix}$$

$$= \frac{1}{21 - 22} \begin{bmatrix} 3 & -11 \\ -2 & 7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3 & 11 \\ 2 & -7 \end{bmatrix}$$

$$x = A^{-1} b$$

$$= \begin{bmatrix} -3 & 11 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \cdot 4 + 5 \cdot 11 \\ 2 \cdot 4 - 7 \cdot 5 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} 43 \\ -27 \end{bmatrix}}$$

<u>Ex.</u> Suppose T is the linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 6x - 5y \\ -6x + 7y \end{bmatrix}.$$

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$A = [T(e_1)\ T(e_2)]$ is size 2×2.

Q_1: Is T one-to-one?

Q_2: Is T onto?

Q_3: Is T invertible? What is T^{-1}?

T is one-to-one if $T(x) = T(y) \Rightarrow x = y$

only one x way to get $T(x) = b$ for any particular b.

T is one-to-one

$\Leftrightarrow$ A_T has a pivot in every col.

$\Leftrightarrow$ cols of A_T are lin ind

$\Leftrightarrow$ $T(x) = b$ has at most one soln for any choice of b.

examples

One-to-one	Not-one-to-one
reflection $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	projection $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
rotation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	zeroing out $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	

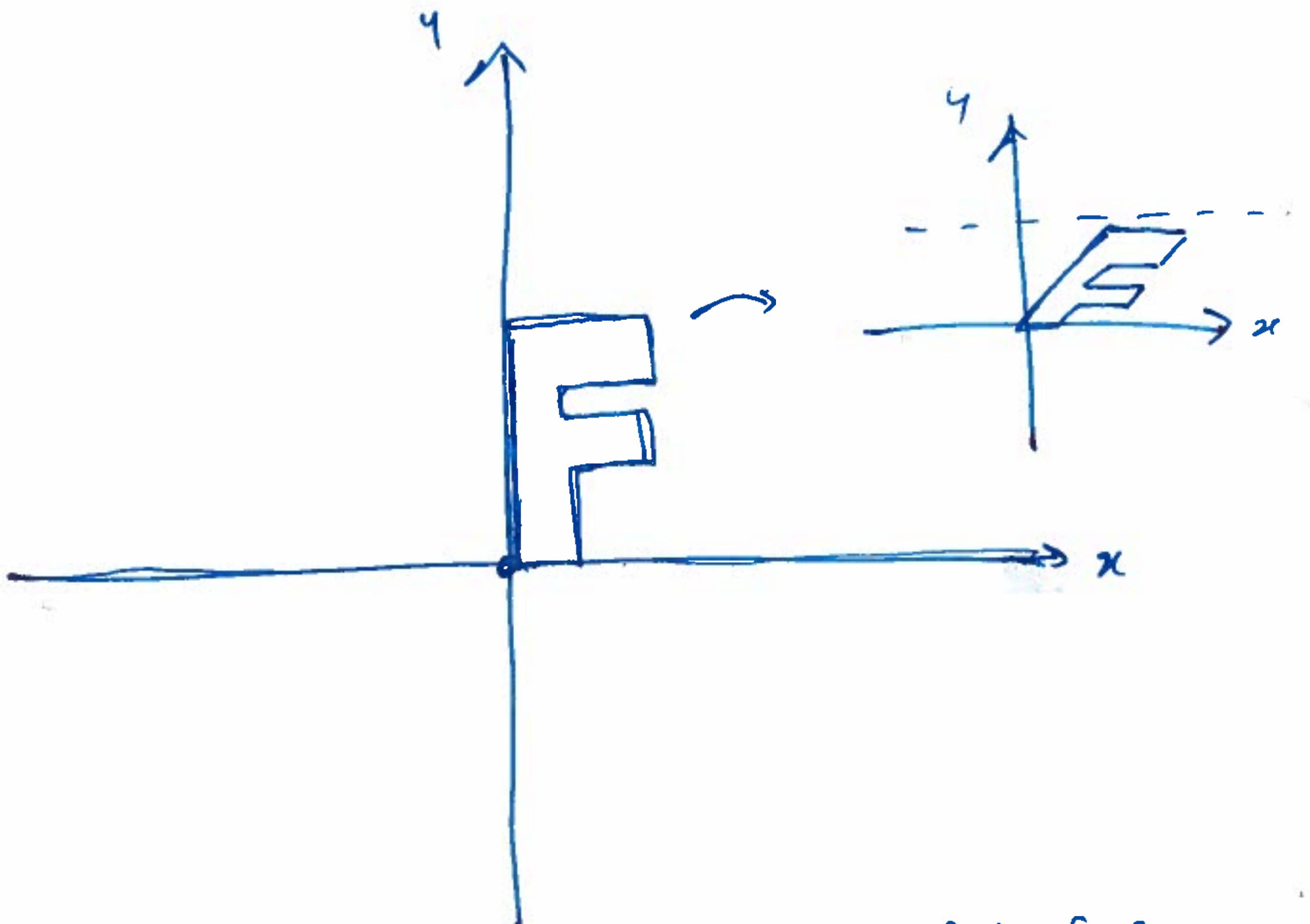
$$A = [T(e_1)\ T(e_2)] = \begin{bmatrix} 6 & -5 \\ -6 & 7 \end{bmatrix}.$$

$A \sim \begin{bmatrix} 6 & -5 \\ 0 & 2 \end{bmatrix}$ 2 pivots $\Rightarrow$ cols lin ind

$\Rightarrow$ T is one-to-one

What is a shear?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is called a } \underline{\text{shear}}.$$



$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 6x - 5y \\ -6x + 7y \end{bmatrix} \leftrightarrow A = \begin{bmatrix} 6 & -5 \\ -6 & 7 \end{bmatrix}$$

2 pivots $\Rightarrow$ T is **1-1**.

Q_2: Is T onto?

Is it possible to solve $T(x) = b$ for any choice of b $\longleftrightarrow$ Is $Ax = b$ always consistent for any choice of b?

Solve?

$$\begin{bmatrix} 6 & -5 \\ -6 & 7 \end{bmatrix} x = b$$

$\boxed{\text{alt. check if A has pivots in every row.}}$

If A then $A^{-1} = \frac{1}{42-30}\begin{bmatrix} 7 & 5 \\ 6 & 6 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 7 & 5 \\ 6 & 6 \end{bmatrix}$

So $X = \frac{1}{12}\begin{bmatrix} 7 & 5 \\ 6 & 6 \end{bmatrix} \cdot b$ it is always possible to solve $Ax = b$

<u>b/c</u> A was invertible.

<u>So</u> Q_3: Is T invertible?

If so what is the standard matrix for T^{-1}?

<u>Ans</u>. T^{-1} standard matrix is A^{-1}

Sanity check.

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}6(1) - 5(2)\\ -6(1) + 7(2)\end{bmatrix} = \begin{bmatrix}-4\\8\end{bmatrix}.$$

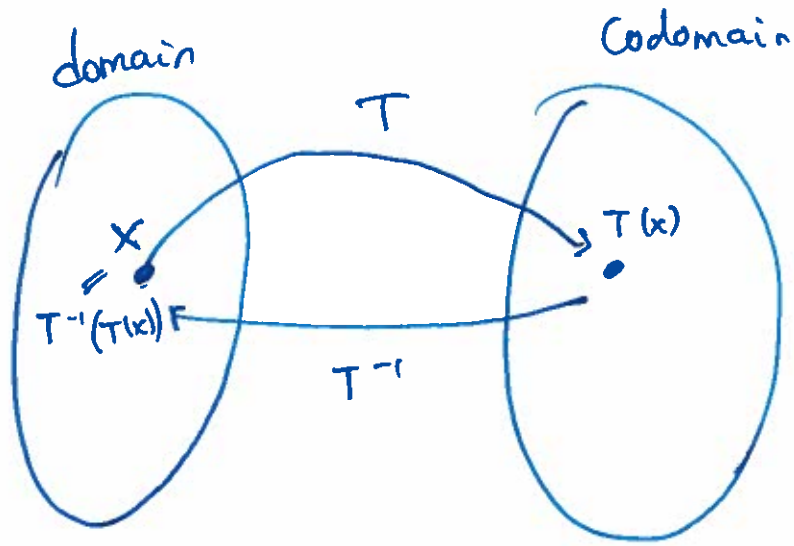
$$T^{-1}\left(\begin{bmatrix}-4\\8\end{bmatrix}\right) = \frac{1}{12}\begin{bmatrix}7 & 5\\ 6 & 6\end{bmatrix}\begin{bmatrix}-4\\8\end{bmatrix} = \frac{1}{12}\begin{bmatrix}-28+40\\ -24+48\end{bmatrix} = \frac{1}{12}\begin{bmatrix}12\\24\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

So $T^{-1} \circ T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}.$

In general

$$\left.\begin{aligned} T^{-1} \circ T(x) &= x \\ T \circ T^{-1}(x) &= x \end{aligned}\right\}$$

What is "∘" circle thing. means "composition"



Next up. LU - Factorization.

Find the LU-factorization for A

$$A = \begin{bmatrix} 12 & 9 \\ 24 & 14 \end{bmatrix}$$

Step 1: Row reduce A to ref

(so that the result is upper-Δ)

$$A = \begin{bmatrix} 12 & 9 \\ 24 & 14 \end{bmatrix} \underset{-2R_1+R_2}{\sim} \begin{bmatrix} 12 & 9 \\ 0 & -4 \end{bmatrix}$$

U

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$ ← what you get if you do $-2R_1+R_2$ to I.

NOTICE

$$E \cdot A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 12 & 9 \\ 24 & 14 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 0 & -4 \end{bmatrix}$$

$$E \cdot A = U \Rightarrow E^{-1} \cdot A = E^{-1} \cdot U$$

$$\Rightarrow A = \underbrace{E^{-1}}_{L} \cdot U$$

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{So } \boxed{L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 12 & 9 \\ 0 & -4 \end{bmatrix}}$$

ANS (type)

$$A = L \cdot U$$

lower triangular

upper triangular.

lower $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ upper $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

upper-Δ

lower-Δ

$$\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

Ex. Find the LU-factorization of

$$A = \begin{bmatrix} 5 & 2 \\ 15 & 1 \end{bmatrix}$$

(and check your answer!)

Step 1: Find U by ~~reduc~~ getting ref of A.

$$A = \begin{bmatrix} 5 & 2 \\ 15 & 1 \end{bmatrix} \underset{-3R_1+R_2}{\sim} \begin{bmatrix} 5 & 2 \\ 0 & -5 \end{bmatrix}$$

this is U

Step 2 Find L.

mult. by -1

always

$$L = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

find this guy

Check $A = L \cdot U$?

$$\begin{bmatrix} 5 & 2 \\ 15 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & 1 \end{bmatrix} \checkmark$$

Why LU at all?

Say you want to solve a "nasty"

$$Ax = b$$

assume easy (or <u>given</u>) to find L, U.

Then

$$Ax = b$$

$$\Leftrightarrow \quad L \cdot U x = b$$

$$\Leftrightarrow \quad L(Ux) = b$$

<u>Step 1</u> Solve $L(y) = b$

this is easy b/c L is lower-triangular.

<u>Step 2</u> Solve $Ux = y$

This is also easy, b/c U is upper-triangular!

(we've replaced 1 hard problem w/ 2 <u>easy</u> problems)

Today 2/22 Week 7

1) A is invertible if there is a matrix "A^{-1}" such that

$$A \cdot A^{-1} = I$$

2) T is invertible if there is a transformation "T^{-1}" such that

$$T \circ T^{-1}(x) = x$$

3) $A \sim \cdots \sim I$ means there are elementary matrices $E_1, E_2, \ldots, E_k$

s.t. the second row operation, the first row operation

$$\underbrace{(E_k \cdots E_2 E_1)}_{A^{-1}} A = I$$

Using
LU-decomp to solve $Ax=b$.

Ex. Find LU & use it to solve $Ax=b$

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}.$$

Soln. I want x such that $Ax=b$.

(one way $[A|b] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$)

$$A \sim \begin{array}{c} \\ 1R_1+R_2 \\ -2R_1+R_2 \end{array} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix} \sim_{5R_2+R_3} \boxed{\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U}$$

$-2R_1+R_2$

$1R_1+R_2$ $\quad$ $5R_2+R_3$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

and.

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}.}$$

Check $A = LU$?

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \checkmark$$

Use $A=LU$ to solve $Ax=b$.

$$Ax=b$$

$$\Rightarrow LU\cancel{x}=b$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \left(\underbrace{\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{"}y\text{"}} \right) = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}.$$

Solve

$$Ly=b$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & -5 & 1 & +16 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Next ~~Solve~~ $y = \begin{bmatrix} -7 \\ -2 \\ +6 \end{bmatrix}$.

$$\boxed{X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 65/3 \\ 4 \\ -6 \end{bmatrix}}$$

Solve $Ux=y$

$$\left[\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & +6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & -7 & 0 & 37 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & 65 \\ 0 & +1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 65/3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

(more practice) 4

Find LU-decomp. (but don't do anything with it.)

$$A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ 2R_1+R_2 \\ 3R_1+R_3 \end{array} \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{bmatrix} \sim \begin{array}{c} \\ \\ 5R_2+R_3 \end{array} \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix}$$

5

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -4 & 3 & -1 \\ 2 & 0 & 0 \end{bmatrix}$$

Something weird will happen with this one.

$$\sim \begin{array}{r} \\ 2R_1+R_2 \\ -R_1+R_2 \end{array} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & +1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\leftarrow$ already upper Δ!

after only dealing w/ first col.

U

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \checkmark$$

$$\underset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \cdot \underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 2 & 0 & 1 \\ -4 & 3 & -1 \\ 2 & 0 & 0 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix}}$$

$$(E_2E_1)^{-1} E_2E_1 A = A = \underbrace{(E_2E_1)^{-1}}_{L} \cdot U$$

and.

$$(E_2 E_1)^{-1} = E_1^{-1} E_2^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \underline{\underline{L}}$$