Today: Dimension, subspaces, and determinant

Recall: A subspace of $\mathbb{R}^n$ is a set of vectors W which satisfy

(1) If $v, w \in W$ then $v + w \in W$

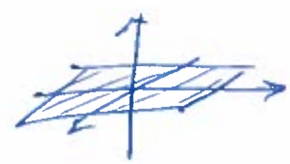
(2) If $v \in W$ and $c \in \mathbb{R}$ then $c \cdot v \in W$.

So subspaces of $\mathbb{R}^n$ are sets of vectors which are

(1) closed under vector addition

(2) closed under scalar mult.

Ex's (these are all subspaces)



* any line through the origin
* any plane through the origin
* any k-plane through the origin
* any span$\{v_1, \dots, v_k\}$
* the range of $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation.
* The solutions to $Ax = 0$, for any A.
* the set of vectors $W \subseteq \mathbb{R}^n$ which map to zero under $T: \mathbb{R}^n \to \mathbb{R}^m$ linear. ("The kernel of T")

Lets check the defn. Pick "A" your fav. matrix

Any x's which satisfy

$$Ax = 0$$

will form a subspace W

called the <u>NULL SPACE</u> of A.

<u>Why is it a subspace?</u>

suppose

x, y both satisfy $Ax = 0$ & $Ay = 0$.

Lets check $A(x+y) \overset{?}{=} 0$

$\Longrightarrow Ax + Ay \overset{?}{=} 0$

$\Longrightarrow 0 + 0 \overset{\checkmark}{=} 0$

} closed under vector add.

$A \cdot (cx) = c \cdot Ax = c \cdot 0 = 0.$ ✓ } Closed under scalar mult.

Ex. Find a <u>basis</u> → spanning set of → lin ind vectors

for the null space

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

<u>Soln.</u> I want vectors which are

① in the null space "$Ax = 0$"

② are lin ind.

③ enough to span "all solns"

Step 1 Find the parametric vector form of solns to $Ax=0$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \underset{\text{(2 steps)}}{\sim} \begin{array}{cccc} & & s & t \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = -s - 2t$
$y = -t$
$z = s$ (free)
$w = t$ (free)

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

these vectors can be chosen as a basis for null of A.

(1) every soln to $Ax=0$ is of the form

$$c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ consists of ALL solns to $Ax=0$.

and these vectors are lin independent.

So they (1) span the null space and are (2) lin ind vectors.

So they are a basis for null of A.

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Q: Joseph says $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \mathcal{B}_1$

S_1

is a basis for range of T

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}.$$

Q: David says $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathcal{B}_2$ is a

S_2

basis for the range of T.

They are both right.

That's why we say $\mathcal{B}_1$, $\mathcal{B}_2$ are each bases.

$\mathcal{B}_1$ is a basis for range of T

$\mathcal{B}_2$ is another basis for range of T.

The dimension of a subspace W is the # of vectors in ANY basis.

The RANK of a matrix is the number of pivots in the rref. (dim of the column space)

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Compute $\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = (1)(4)-(2)(3)$

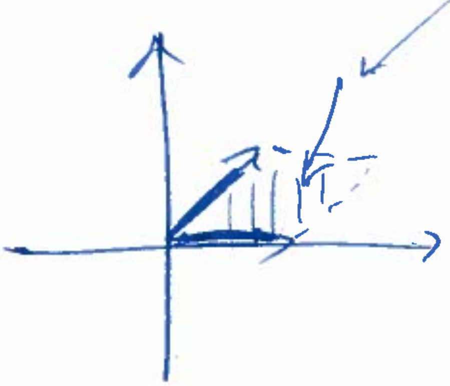
$= 4-6 = \boxed{-2}$

$\det\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \boxed{1}$

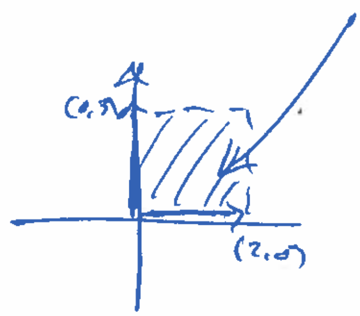
use formula

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = ad-bc.$$

$\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 1$

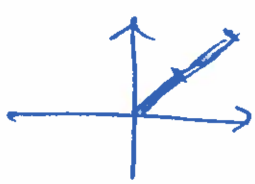


$\det\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix} = 6$



The det of A is the volume of the parallelpiped formed by the columns of A.

$\det\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} = 0$



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$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - d \cdot \det\begin{bmatrix} b & c \\ h & i \end{bmatrix} + g \cdot \det\begin{bmatrix} b & c \\ e & f \end{bmatrix}.$$

for +/- choices

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & -2 \\ 3 & 1 & 3 \end{bmatrix}$$

[Cofactor expansion]

cofactors.

using col 1.

$$= 1 \cdot \det\begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix} - 2 \cdot \det\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} + 3 \cdot \det\begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix}$$

$$= 1 \cdot (12 + 2) - 2(-1) + 3(-4)$$

$$= 14 + 2 - 12 = 4$$

using row 1

$$= 1 \cdot \det\begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix} - 0 \cdot \det\begin{bmatrix} 2 & -2 \\ 3 & 3 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$= 14 - 0 - 10 = 4$$

Compute

$$\det \begin{bmatrix} a & * & * & * & * \\ 0 & b & * & * & * \\ 0 & 0 & c & * & * \\ 0 & 0 & 0 & d & * \\ 0 & 0 & 0 & 0 & e \end{bmatrix} = a \cdot b \cdot c \cdot d \cdot e$$

$$= a \cdot \det \begin{bmatrix} b & & & \\ & c & & \\ & & d & \\ 0 & & & e \end{bmatrix} + 0 \;—$$

$$= a \cdot \left(b \cdot \det \begin{bmatrix} c & & \\ 0 & d & \\ 0 & 0 & e \end{bmatrix} \right)$$

$$= a \cdot b \cdot c \cdot \det \begin{vmatrix} d & * \\ 0 & e \end{vmatrix}$$

$$= \boxed{abc \cdot de}$$

week 8 wed 3/1

Today:

* abstract vector spaces (15 min)
  & subspaces
* examples of finding bases
  & basis review/exploration

Defn. An abstract vector space is a set X equipped with two operations called vector addition and scalar multiplication such that

if 1) $v, w \in X$ then $v+w \in X$

2) If $v \in X$ and c is a scalar then $cv \in X$.

Consequences:

1) $0 \in X$ always

"$\subseteq$" subset

"$\{\ast\}$" set

2) X is a subspace of itself

3) $\{0\} \subseteq X$ is a subspace called the "trivial subspace"

Ex. $\mathbb{R}^n$ is a abstract vector space (here are two more...)

The set of

#1. Polynomials with usual $+, *$. $\sqrt{2}$

1) If f is a poly, and g is a poly then $f+g$ is a poly. (Closed under "$+$").

2) If f is a poly, c is a scalar then cf is a poly too (closed under "$*$")

What are some subspaces of $X = \{polys\}$?

$$W = \{\text{poly's of degree} \leq 3\} \subseteq X$$

↑ is a subspace of X.

e.g. $x^3 + 2x^2 - 1$, $3x^2 + 2x + 4$

Q: What is a basis of W?

want: $\{v_1, v_2, \ldots, v_k\}$ a set of vectors which are lin ind & they span W.

$$\mathcal{B} = \{x^3, x^2, x, 1\}$$

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$.

NOTE $\text{span}\{x^3, x^2, x, 1\} = \{ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R}\}$
$= W \approx \mathbb{R}^4$

#1 (non-example)

$X = \{\text{polys}\}$

$W = \{\text{poly's of degree} = 3\}$

<u>Note.</u> W "has a basis" (sort of kinda)

$\text{span}\{1, x, x^2, x^3\} \supseteq W$

are in not

W is <u>not a subspace</u> b/c

e.g. $(\underbrace{x^3 + 2x + 1}_{\text{deg } 3}) + (\underbrace{-x^3 + 3x^2 + 1}_{\text{deg } 3})$

$= -3x^2 + 2x + 2$ deg 2 !!

not closed under vector "+".

e.g. $0 \cdot f = 0 \leftarrow$ not of degree 3.

not closed under scalar "$*$".

<u>FACT</u>: all subspaces contain the vector $0 \in W$.

#2. $\{m \times n \text{ matrices}\} = X$

is an abstract vector space for any choice of m, n.

Q: $X = \{3 \times 3 \text{ matrices}\}$

What is a basis for X?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots + i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The dimension of X is 9.

$X \approx \mathbb{R}^9$ (isomorphic: There is a $T: X \to \mathbb{R}^9$ linear transformation which is 1-1 & onto)

In general

$X = \{m \times n \text{ matrices}\} \approx \mathbb{R}^{m \cdot n}$

5.

Ok, now back to reality.

Q_1: Find a basis for nul(A)

Q_2: Find a basis for col(A)

Q_3: compute rank(A) & nullity(A).

$$A = \begin{bmatrix} 2 & 0 & 4 & 8 \\ -1 & 0 & -2 & -4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \leftarrow \begin{cases} 2x + 4z + 8w = 0 \\ -x - 2z - 4w = 0 \\ 2z + 8w = 0 \end{cases}$$

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recall: $\text{nul}(A) = \{x : Ax = 0\}$

$= \{\text{all vectors } x \text{ which satisfy } Ax = 0\}$

this is a subspace: if $Ax = 0$, $Ay = 0$ then $A(x+y) = Ax + Ay = 0 + 0 = 0$

also if $Ax = 0$ then $A \cdot (cx) = c\,Ax = 0$

Soln. (Q_1) Find solns to $Ax = 0$, get parametric vector form.

$$A \sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} & t & & s \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \leftarrow \boxed{\begin{array}{l} x - 4w = 0 \\ z + 4w = 0 \end{array}}$$

$\leftarrow$ rref of A.

$x = 4s$
$y = t$ (free)
$z = -4s$
$w = s$ (free)

$$\longrightarrow \quad X = \begin{bmatrix} 4s \\ t \\ -4s \\ s \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

ANS

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}}$$

(basis for col A?)

(Q2)

Soln.

$$A = \begin{bmatrix} 2 & 0 & 4 & 8 \\ -1 & 0 & -2 & -4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \right\}$.

(Q3) rank & nullity.

Soln.

$$\text{rank}(A) \overset{\text{defn}}{=} \text{dim of col}(A) \overset{\text{Facts}}{=} \#\text{ pivots}$$

$$\text{nullity}(A) = \text{dim of nul}(A) = \#\text{ free vars}$$

$\text{rank}(A) = 2$ and $\text{nullity}(A) = 2$ (for this example)

In general:

THM $\text{rank}(A) + \text{nullity}(A) = \#\text{ cols of } A$.

Proof. every col is either a pivot col. or a free var col. ∎

Q:

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the "floor of $\mathbb{R}^3$". 7

Q_1: What is the standard matrix A of T?

Q_2: What is a basis for range of T?

Q_3: rank A, nullity A?

Sol 1)

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Soln 3) rank $A = 2$ nullity $A = 1$.

Soln 2) <u>recall</u> range of T = col(A)

= span of cols of A

= Span $\{e_1, e_2\}$

= "the floor"

The "floor" has many bases but one is $\{e_1, e_2\}$.

$$\text{col}(A) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

a basis is $\{e_1, e_2\}$.