Today:

* coordinates
* properties of determinants

Q: What are the coordinates of

$b = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ in the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \end{bmatrix} \right\}$?

IF $$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Then the coordinates of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ in the basis $\mathcal{B}$ are $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\mathcal{B}}$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 1 & -5 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & -3 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 5/3 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3+2\cdot(5/3) \\ 0 & 1 & 5/3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 19/3 \\ 0 & 1 & 5/3 \end{array}\right].$$

$$\frac{19}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$ Check? ✓

Q: What are the coordinates of

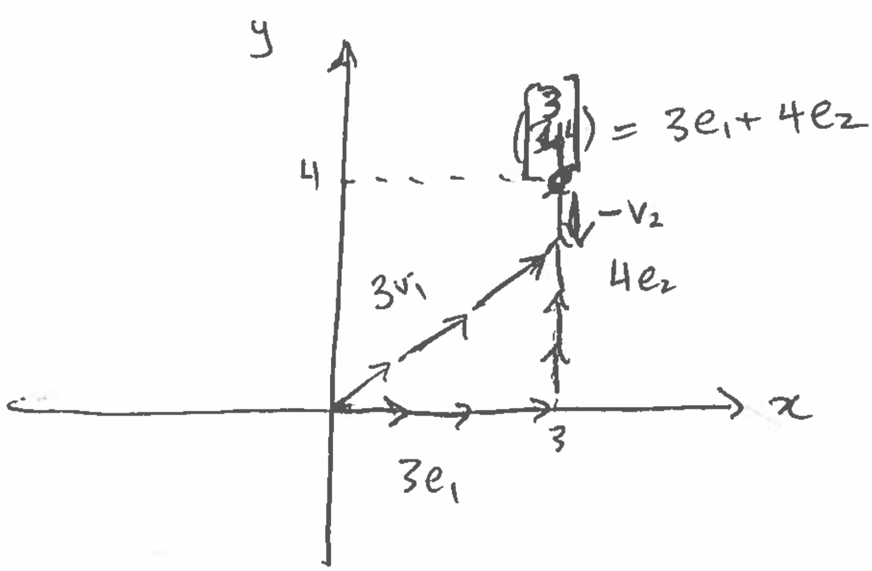
$\begin{bmatrix}1\\2\\3\end{bmatrix}$ in the standard basis $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$?

$$c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

visually, by inspection

$$c_1 = 1, \quad c_2 = 2, \quad c_3 = 3.$$

coord. $\begin{bmatrix}1\\2\\3\end{bmatrix}_{\mathcal{B}}$ are the <u>usual</u> coordinates.



In the basis $\{e_1, e_2\}$ the vector $\begin{bmatrix}3\\4\end{bmatrix}$'s coordinates are "$\begin{bmatrix}3\\4\end{bmatrix}$"

in $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix} \right\}$ $\begin{bmatrix}3\\-1\end{bmatrix}_{v_1, v_2}$

$$3\begin{bmatrix}1\\1\end{bmatrix} + (-1)\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$$

Q: What are the standard coordinates

of the vector $\begin{bmatrix} 5 \\ -3 \end{bmatrix}_{\mathcal{B}}$ where

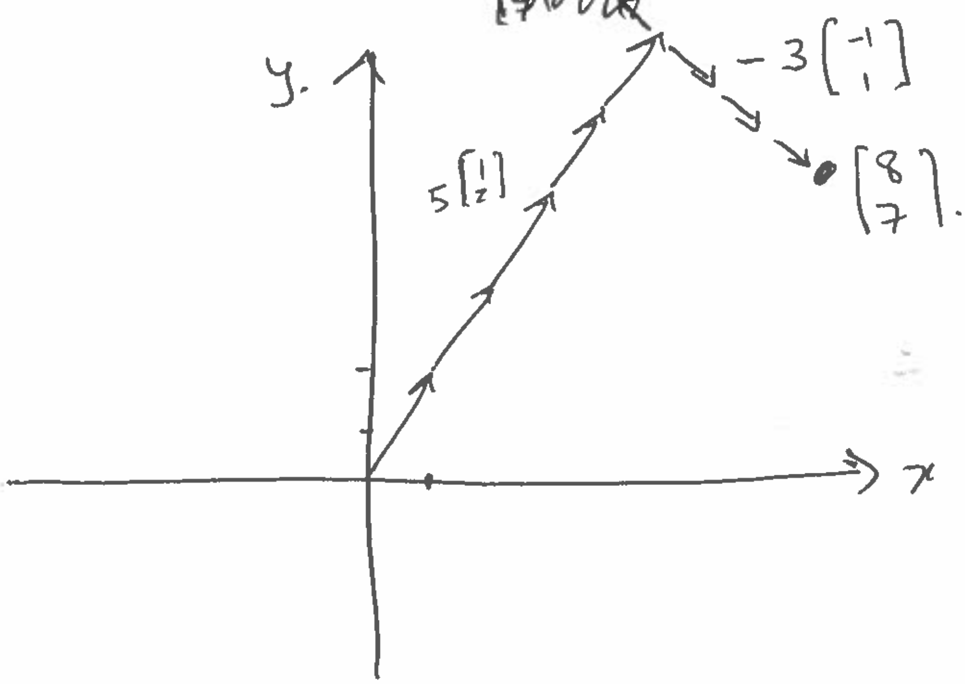
"recipe"

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$?

"ingredients"

$$5\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5+3 \\ 10-3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$



Facts about determinants

1) det is <u>multiplicative</u> for $n \times n$ matrices A, B.

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

e.g.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

So.

$$\det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = 2$$

can be computed as

$$\det\left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot \det\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

$$= 1 \cdot (4-2) = 1 \cdot 2 = 2 \quad \checkmark$$

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2) determinant is <u>NOT</u> additive

$$\det(A+B) \neq \det(A) + \det(B) \quad \text{in } \underline{\text{general}}.$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\det(\cdot) = 1} + \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}_{\det(\cdot\cdot) = -2} = \underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}}_{\det(\cdot\cdot) = 0}.$$

3) If A is an $n \times n$ matrix w/ two (or more) rows identical, then $\det(A) = 0$.

4) How row operations effect determinants.

Let A be square $n \times n$ matrix.

then

1) switch two rows of A $\longrightarrow$ $\det(A) * -1$.

e.g. $\det\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = 2(4) - (1)3 = 5$

$\det\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = (3)(1) - (4)(2) = -5$

2) Scale a row by c $\longrightarrow$ $\det(A) * c$

$\det\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = 5$

$\det\begin{bmatrix} 20 & 10 \\ 3 & 4 \end{bmatrix} = 20(4) - 10(3) = 50$

3) add $cR_i + R_j \rightarrow R_j$ $\longrightarrow$ $\det(A)$ is unchanged.

$\det\begin{bmatrix} 2 & 1 \\ 0 & 5/2 \end{bmatrix} = 2(5/2) - 1(0) = 5$

$-\frac{3}{2}R_1 + R_2 \rightarrow$

EX. Compute $\det(A)$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

#1 option use cofactor expansion

col 2

$$\det A = -0 + 0 - 1 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -(-1) = 1.$$

row 3

$$= 4 \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}$$

$$= 4 \cdot 0 - 1(-1) + 2 \cdot 0 = 1. \checkmark$$

#2 option use row operations

type 3 only.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 4 & 1 & 2 \end{bmatrix} \underset{-4R_1+R_3}{\overset{-2R_1+R_2}{\sim}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}}$$

These didn't change the det

$\det(\text{''}) = \underline{\underline{-1}}.$

This one mult. by -1.

So $\det(A) \cdot -1 = -1$ so $\boxed{\det(A) = 1}$

Notice we did 3 row operations to A $E_3 E_2 E_1 A = B$.

$$\underset{R_2 \leftrightarrow R_3}{\overset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}}
\underset{-4R_1 + R_3}{\overset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}}
\underset{-2R_1 + R_2}{\overset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}
\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

Since $E_3 E_2 E_1 A = B$

then $\det(E_3 E_2 E_1 A) = \det(B)$

since B is upper Δ it's easy.

$$= \underset{-1}{\det(E_3)} \cdot \underset{1}{\det(E_2)} \cdot \underset{1}{\det(E_1)} \cdot \det(A) = \det(B) = -1$$

$$-\det(A) = -1$$

So $\det(A) = 1$.