Week 10 Monday. 3/13

Defn. An eigenvector $\underline{x}$ of A is a non-zero vector $x \neq 0$ such that

$$Ax = \lambda x \quad \text{for some } \lambda \in \mathbb{R}.$$

← scalar λ.

(λ is called an eigenvalue associated to eigenvector $\underline{x}$)

e.g. $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Is u an eigenvector of A?

$$A \cdot u = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 - 30 \\ 30 - 10 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix}.$$

Is $A \cdot u = \lambda u$ for some good choice of λ?

$$\begin{bmatrix} -24 \\ 20 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad ?$$

Yes $\lambda = -4$.

So $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector of A w/ associated eigenvalue λ.

Next. What about $v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$?

$$A \cdot v = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-12 \\ 15-4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$

is $\begin{bmatrix} -9 \\ 11 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ for some λ?

<u>No.</u> Since $-9 = (-3)(3)$ but $11 \neq (-3)(-2)$ so no λ will work.

(in particular $\begin{bmatrix} -9 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are lin. ind.)

Q: What do eigenvectors "look like"?

Consider $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ and try to find some eigenvectors.

$$\boxed{T_A(x) = \lambda \cdot x}$$

1) T_A = rotation by $180°$

$T_A(v) = A \cdot v = -v$

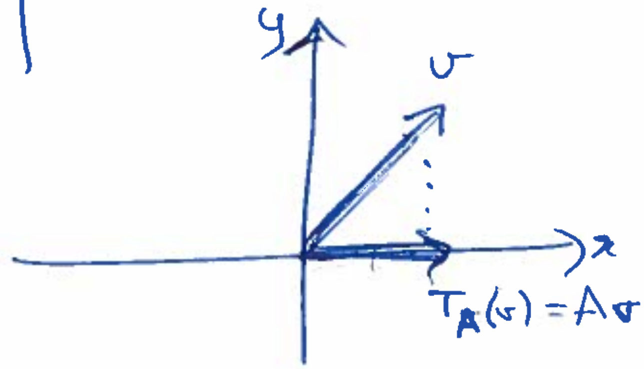
by the way $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

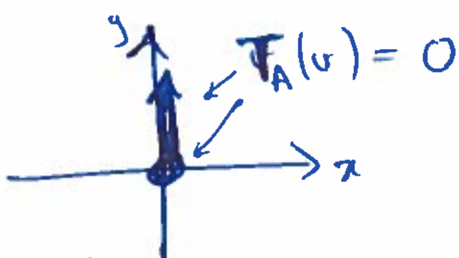
eigenvalue $\lambda = -1$ for <u>every</u> $x \in \mathbb{R}^2$

nextup... T_A = projection to x-axis

$T_A : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$



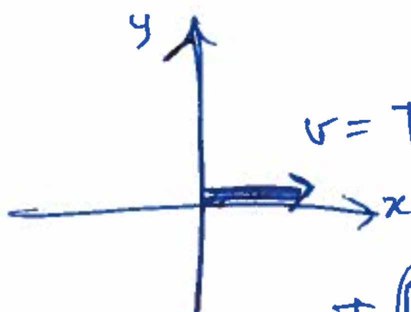


$T_A\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\right) =$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ w/ ass. eigenvalue $\lambda = 0$.

on the x-axis.



$v = T_A(v)$ if v on x-axis.

$T_A\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) =$

e.g. $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

So $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ w/ eigenvalue $\lambda = 1$.

FACT #1 The null space of A consists entirely of $\lambda = 0$ eigenvectors.

FACT #2 geometrically, eigenvectors "stay on the same line" as their image.

Q: How does you find eigenvectors?

Idea: want $X \neq 0$ such that

$$A \cdot x = \lambda \cdot x \quad \text{for some } \lambda.$$

$$\Longleftrightarrow \quad A \cdot x - \lambda \cdot x = 0 \quad \text{for some } x \neq 0.$$

$$\Longleftrightarrow \quad (A - \lambda \cdot I) \cdot x = 0 \quad \text{for some } x \neq 0.$$

This is only possible if

$A - \lambda I$ has a non-trivial soln to

$$(A - \lambda I) x = 0.$$

So it must be non-invertible.

$\boxed{\det(A - \lambda I) = 0}$. if λ is an eigenvalue.

We just saw that $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ had $\lambda = -4$.

Let's check that

$$\det(A - (-4) I) = 0.$$

$$\det\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$$

Q: But how do you FIND THEM?

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

Step 1: compute $\det(A - \lambda I)$ where λ is a "variable"

$$\det\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda) - 6\cdot 5$$

$$= \lambda^2 - 3\lambda + 2 - 30$$

$$= \lambda^2 - 3\lambda - 28$$

$$= (\lambda+4)(\lambda-7)$$

Step 2: Set $\det(A - \lambda I) = 0$ & solve for λ

$\det(A - \lambda I) = (\lambda+4)(\lambda-7) = 0$ iff $\underline{\underline{\lambda = -4, 7}}$.

Step 3: Solve $Ax = \lambda \cdot x$ for each choice of λ.

Solve for x. $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

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$$A \cdot x = 7 \cdot x$$

(i.e. find null space of $A - 7I$)

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 7 \cdot \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\Bigg(\underbrace{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}}_{A} - \underbrace{7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{7I} \Bigg) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector w/ eigenvalue $\lambda = 7$

$$A - 7I = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad x = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Check. Supposed to have.

$$A \cdot x = 7 \cdot x \quad \text{iff} \quad x = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Check for $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \checkmark$$

practice questions.

Q_1: Is $\lambda = 2$ an eigenvalue of $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$?

(justify always)

Q_2: Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$

Q_3: Is $\lambda = 4$ an eigenvalue of $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$?

Q_4: If yes, find an eigenvector.

A_1. Is there $x \neq 0$ s.t. $Ax = 2 \cdot x$? $\leftarrow (A - 2I)x = 0.$

compute $\det(A - 2I) = \det\left(\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right)$

$= \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \right) = (1)(6) - (2)(3) = 0.$

Since $\det(A - 2I) = 0$ for $\lambda = 2$, $\lambda = 2$ is an eigenvalue.

A_2. Check if $A \cdot x = \lambda x$ for some λ.

$\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$

So. <u>No</u> $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ not an eigenvector of A.

FACT Characteristic polynomial.

$p(\lambda) = \det(A - \lambda I) = 0$

iff λ is an eigenvalue of A

$\underline{A_3}$ $\det(A-4I) = \det\left(\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

$= \det\begin{bmatrix} -1 & \boxed{0} & -1 \\ 2 & \boxed{-1} & 1 \\ -3 & \boxed{4} & 1 \end{bmatrix}$

$= -0 + (-1)\det\begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} - 4\cdot\det\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

$= -(-1-3) - 4(-1+2)$

$= +4 - 4(+1) = 0$ ✓

$\underline{A_4.}$ $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$ know $\lambda = 4$ eigenvalue.

<u>WANT</u> $x \neq 0$ eigenvector corr. to $\lambda = 4$.

<u>WANT</u>

$\text{null}(A - 4I)$

$$A - 4I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{array}{c} \quad \quad s \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

punchline.

$x = s\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. | $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector w/ eigenvalue $\lambda = 4$