Welcome back!! Monday 3/27 1

Week 11.

I hope everyone had an excellent SPRING BREAK!...

... OK Now BACK TO WORK!!

This week DIAGONALIZATION!

Idea If A is square, $n \times n$ and has

n linearly independent eigenvectors

Then A is diagonalizable and can be written as

diagonal $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

$$A = PDP^{-1}$$

invertible matrix whose columns are the n linearly indep. eigenvectors of A

eigenvalues λ_i ass. to eigenvector v_i

$$P = [v_1 \; v_2 \; \ldots \; v_n]$$

So.

$$Av_i = \lambda_i v_i$$

<u>Ex.</u> Diagonalize

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

ans

$$= P \cdot D \cdot P^{-1}$$

tell me P tell me D.

DON'T have to compute P^{-1}.

<u>Step 1</u>: Find D.

The entries of the diagonal matrix D are just the eigenvalues of A w/ associated multiplicities.

To find the 3 eigenvalues of A we need to solve the characteristic poly

$$p(\lambda) = \det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 2 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3-\lambda & 1 \end{vmatrix}$$

$$= (2-\lambda)\left[(3-\lambda)(2-\lambda) - 2\right] - \left[2(2-\lambda) - 2\right] + \left[2 - (3-\lambda)\right].$$

idea: keep as factored as possible

$$= (2-\lambda)\left[\lambda^2 - 5\lambda + 4\right] - \left[-2\lambda + 2\right] + \left[-1 + \lambda\right]$$

$$= (2-\lambda)\left[(\lambda-4)(\lambda-1)\right] + 2\left[\lambda - 1\right] + \left[\lambda - 1\right]$$

$$= (\lambda-1)\left[\underbrace{(2-\lambda)(\lambda-4) + 2 + 1}_{\text{deg 2 so I can solve quadratics easily!}}\right]$$

recall.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= (\lambda-1)\left[-\lambda^2 + 6\lambda - 5\right]$$

$$= (\lambda-1)\left(-\left[\lambda^2 - 6\lambda + 5\right]\right)$$

$$= -(\lambda-1)(\lambda-5)(\lambda-1) = 0$$

$\lambda = 1, 5$

mult. 2 mult. 1.

So $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Step 2: For each λ you have to find "the right number" of eigenvectors.

as many as the alg. mult. of λ.

$\lambda = 5$ $\quad \text{null}(A - 5I) = \text{null}\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$

$$A - 5I \sim \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = s\begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}$ ← basis for the $\lambda = 5$ eigenspace

Next, we want the 2 eigenvectors for eigenvalue $\lambda = 1$, had algebraic multiplicity 2

(Note: If we don't find $\underline{\underline{2}}$ lin. ind. $\lambda = 1$ eigenvectors, then the matrix A is $\underline{\text{NOT}}$ diagonalizable)

$\underline{\underline{\lambda = 1}}$

$$\text{null}(A - I) = \text{null}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \; ; \; A - I \sim \begin{array}{ccc} & s & t \\ \end{array} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x + 2s + t = 0$
$y = s$ (free)
$z = t$ (free)

$\left.\right\}$ $x = -2s - t$
$y = s$ (free)
$z = t$ (free)

$$X = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$X = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Step 3: Construct P.

$$\boxed{ P \overset{①}{=} \begin{array}{ccc} \lambda=1 & \lambda=1 & \lambda=5 \\ \end{array} \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D \overset{②}{=} \begin{array}{ccc} \lambda=1 & \lambda=1 & \lambda=5 \\ \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad A = PDP^{-1} \; ③ }$$

$\underline{\underline{\text{OR}}}$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or any comb. which keeps $\lambda_i \leftrightarrow v_i$ (right ...)

<u>Kicker</u>: Use diagonalization to compute

$$A^n = (PDP^{-1})^n$$

$$= \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\ldots(\cancel{P}DP^{-1})}_{n \text{ times}}$$

$$= P \cdot D^n \cdot P^{-1}$$

<u>punchline</u>

$$D^n = \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_k^n \end{bmatrix}$$

The power of a diagonal matrix is diagonal & the entries are the powers of the original matrix.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad D^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \quad D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$$

Ex. Compute A^4 where

$$A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}.$$

Step 1: diagonalize A. $A = P \cdot D \cdot P^{-1}$

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{bmatrix}$$

$$= +(\lambda+2)(\lambda-5) + 12$$

$$= +(\lambda^2 - 3\lambda - 10) + 12$$

$$= +\lambda^2 - 3\lambda + 2$$

$$= (\lambda - 2)(\lambda - 1)$$

$\lambda = 1, 2$ (both have mult. 1)

$\lambda = 1$: $\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ $\quad x = s\begin{bmatrix} +4 \\ 1 \end{bmatrix}$.

$\lambda = 2$: $\begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\quad x = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad P = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$.

Step 2 : Compute $A^n = P \cdot D^n \cdot P^{-1}$

$$A^4 = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^4 \cdot \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \cdot \overset{1}{\cancel{\frac{1}{4-3}}} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 48 \\ 1 & 16 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -44 & -12 + 4(48) \\ -15 & -3 + 4(16) \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -44 & 180 \\ -15 & 61 \end{bmatrix}}$$

3/29

Wed

week 11

Some fun (theoretical) facts about eigenvalues and diagonalization.

#1 The set of ~~eigenvalues~~ eigenvectors of A associated to a particular eigenvalue λ is a subspace of $\mathbb{R}^n$ (if A is $n \times n$) called the eigenspace of A.

Proof. $V_\lambda = \{x : Ax = \lambda x\}$ the eigenspace is just $\text{null}(A - \lambda I)$, which is a subspace.

#2 If $\lambda = 0$ is an eigenvalue of A, then the $\lambda = 0$ eigenspace is just $\text{null}(A)$. In particular, A is not invertible! (and visa versa).

Proof. $Ax = 0 \cdot x$, $x \neq 0 \iff A$ not invertible.

2

#3. The algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dim of the corr. eigenspace, that is, the number of linearly independent eigenvectors for that eigenvalue.

alg. mult $\geq$ geometric mult.

# of times λ is a root of $p(\lambda)$. the characteristic poly.

# of linearly ind eigenvectors for λ.

#4. If $\lambda_1 \neq \lambda_2$ are both eigenvalues, but distinct (different)
then if v_1 eigenvector for λ_1,
v_2 eigenvector for λ_2,

then v_1, v_2 are linearly independent.

Proof. Suppose $\lambda_1 \neq \lambda_2$ two distinct eigenvalues w/ eigenvectors v_1, v_2 resp. Suppose v_1, v_2 are linearly dependent. ~~Then~~ Then $\boxed{c v_1 = v_2}$ for some scalar $c \neq 0$

mult by A: $A \cdot (c v_1) = A v_2$

$c \cdot (A v_1) = A v_2$

mult. by λ_2: $\lambda_2 \cdot c v_1 = \lambda_2 v_2$

$c \cdot \lambda_2 v_1 = \lambda_2 v_2$

So we have both are true:

$$c \cdot \lambda_1 v_1 = \lambda_2 v_2 \quad \& \quad c \cdot \lambda_2 v_1 = \lambda_2 v_2$$

subtract the eqns from each other.

$$c \cdot \lambda_1 v_1 - c \cdot \lambda_2 v_1 = \lambda_2 v_2 - \lambda_2 v_2 = 0$$

So.

$$c \cdot (\lambda_1 - \lambda_2) \cdot v_1 = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0 \Rightarrow \boxed{\lambda_1 = \lambda_2} \;\; \cancel{\alpha}$$

"contradiction"

This is true in general.

Suppose $\lambda_1, \lambda_2, \lambda_3$ are all eigenvalues, all distinct, and v_1, v_2, v_3 are eigenvectors ass. to $\lambda_1, \lambda_2, \lambda_3$ (respectively).

Then v_1, v_2, v_3 are lin. ind.

Proof: Suppose not. We already saw that v_2 is not a scalar mult. of v_1 (from previous example)

So $c_1 v_1 + c_2 v_2 = v_3$ (otherwise can rename v_1, v_2, v_3) $c_1, c_2 \neq 0$.

mult. by A.

$$A c_1 v_1 + A c_2 v_2 = A v_3$$

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = \lambda_3 v_3$$

mult. by λ_3

$$\lambda_3 c_1 v_1 + \lambda_3 c_2 v_2 = \lambda_3 v_3$$

$$c_1 \lambda_3 v_1 + c_2 \lambda_3 v_2 = \lambda_3 v_3$$

$$c_1(\lambda_1 - \lambda_3) v_1 + c_2(\lambda_2 - \lambda_3) v_2 = 0$$

v_1, v_2 are lin ind so $\lambda_1 = \lambda_3$ &

Punchline: eigenvectors coming from different eigenvalues are linearly independent. (#4)

In order to determine whether or not a matrix $n \times n$ is diagonalizable you have to decide if there are n linearly independent eigenvectors.

ONE WAY: If there are n distinct eigenvalues, then A is diagonalizable.

eigenvectors

$$A = PDP^{-1}$$

eigenvalues

OTHER WAY: If not all eigenvalues are distinct, then you have to check that

alg mult. = geo. mult.

for each eigenvalue (w/ mult. > 1)

Q: Is it diagonalizable?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad ?$$

Ans: yes. Notice

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

$$= (1-\lambda)(4-\lambda)(6-\lambda) = 0$$

get $\lambda = 1, 4, \text{ or } 6$. all distinct.

Q: Is $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ diagonalizable?

yes. $A = P \cdot D \cdot P^{-1}$

diagonal

invertible

choose $P = I$, $D = A$.

6

Q: Is this diagonalizable

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}. \quad ?$$

Idea: First find all eigenvalues w/ mult.

then decide if

alg mult. = geo. mult (for each λ).

# of times λ is a root of $p(\lambda)$

dim of null$(A-\lambda I)$

or n - rank $(A-\lambda I)$

or # of lin ind eigenv.

$$\det \begin{bmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{bmatrix} = (5-\lambda) \begin{vmatrix} 4-\lambda & 0 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)[(4-\lambda)(5-\lambda) - 0]$$

$$= -(\lambda-5)^2(\lambda-4) = 0$$

$$\lambda = 4, \text{ or } 5.$$

Only care to check $\lambda = 5$ (mult. 2)

$$A - 5I = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$V_5 = \{\lambda = 5$ eigenspace$\}$ has dim 2

geo mult of $\lambda = 5$.

So $\boxed{A \text{ diagonalizable!!}}$