Today: 4/3 Week 12 Monday.

Complex eigenvalues & eigenvectors

So far, we've been working over the

real numbers. $\mathbb{R}$, so our

matrices have entries like

$0, 1, -3, 1/2,$ or even $\pi, \sqrt{2}, \ldots$ etc.

(real numbers are any number with a decimal expansion, if you like.)

Loose end: Two matrices A, B are similar

if for some invertible matrix P,

interesting fact. A, B are similar if they are the same trans. acting on different bases.

$$A = P \cdot B \cdot P^{-1}$$

So A is similar to B only if B is similar to A. (and visa versa).

Note this means

$$B = (P^{-1})(A) \cdot (P^{-1})^{-1}$$

(defn). A is diagonalizable if it's similar to a

We may sometimes want to solve for eigenvalues & eigenvectors over $\mathbb{C}$.

The complex numbers

Ex. $i = \sqrt{-1}$ so that's the number w/

$$(i)^2 = -1.$$

$2i$, $1+i$, $1-i$, $3+2i$, $4(3+2i)$, $(1+i)(3+2i)$

are all examples of complex numbers,

Also just for bookkeeping, all $\mathbb{R}$ numbers are considered to be $\mathbb{C}$ numbers (w/ no imaginary part)

$$(3+2i) + (2-4i)$$

real part imaginary part.

$$= 5 - 2i$$

add up real parts add up imaginary parts

Also, you can multiply $\mathbb{C}$-numbers

$$(3+2i)(2-4i) \overset{?}{=} \underbrace{\quad}_{\text{real part?}} + \underbrace{\quad}_{\text{imaginary part}} i$$

$$6 - 12i + 4i - 8i^2$$

$$= 6 - 8i - 8(-1)$$

$$= \boxed{14 - 8i}$$

Please write complex numbers (in your answers) in the form $a + bi$, $a, b \in \mathbb{R}$.

↑ Standard form of a complex number.

Now we know WHAT complex numbers are.

Next up: WHY? and How?

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{-4}}{2}$$

$$= \frac{\pm\sqrt{-4}}{2} = \frac{\pm 2i}{2} = \boxed{\pm i}$$

Q: Why use $\mathbb{C}$-numbers.

Find all eigenvalues.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = (0-\lambda)^2 - (-1)$$

Solve $= \lambda^2 + 1 = 0$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which is T_A : rotate by 90° counter-cc

$\underline{\underline{\text{has}}}$ $\underline{\underline{\text{no}}}$ $\underline{\underline{\text{real}}}$ eigenvalues.

But it has two complex eigenvalues

$$\lambda = i \quad \text{and} \quad \lambda = -i$$

$\underline{\text{NOTE}}$ complex eigenvalues $\underline{\underline{\text{ALWAYS}}}$ come in conjugate pairs.

if $\lambda = a+bi$ is one, $\lambda = a - bi$ is the other

$\underline{\underline{\text{How?}}}$ to find them, use quadratic formula.

to find the corresponding $\underline{\text{eigenvectors}}$ you have to "do algebra over $\mathbb{C}$"

$+, *, -, \div$

$(a+bi)(a-bi) = a^2 - b^2 i^2$
$= a^2 + b^2$

$$\frac{(3+2i)}{(2-4i)} \cdot \frac{(2+4i)}{(2+4i)} = \frac{6+16i-8}{4+16} = \frac{-2+16i}{20} = \boxed{\frac{-1}{10} + \frac{4}{5} i}$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ Find the eigenvectors for $\lambda = \pm i$.

$\lambda = -i$

$$A - \lambda I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we want the null space of $A - \lambda I$.

$$= \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \sim_{-iR_1 + R_2} \begin{bmatrix} 1 & i \\ 0 & \text{[illegible]} \end{bmatrix}$$

$i(-i) = 1$

$-(-1) - 1 =$

So we need $X = \begin{bmatrix} x \\ y \end{bmatrix}$ so that $\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (s)

$x + iy = 0$
$y =$ free

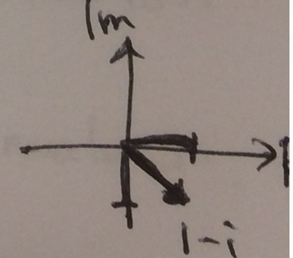
$x = -i \cdot s$
$y = s$ (free)

$$X = s\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Check $A \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -i \\ 1 \end{bmatrix}$?

λ ?

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = \lambda \begin{bmatrix} -i \\ 1 \end{bmatrix}$$ $\vec{x} = 1 - i$



Next. For $\lambda = i$

$$A - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Notice

$\lambda = -i$ $\qquad X = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

replace $a+bi$ by $\overline{a+bi} = a-bi$ "conjugate bar"

$\lambda = i$ $\qquad X = \begin{bmatrix} i \\ 1 \end{bmatrix}.$

replace any z $\mathbb{C}$-number w/ $\overline{z}$'s. (their conjugate).

$A = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}$ Find ALL eigenvectors & eigenvalues.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 5 \\ -5 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 25 = \lambda^2 - 6\lambda + 9 + 25$$

$$= \lambda^2 - 6\lambda + 34 = 0 \qquad \frac{6 \pm \sqrt{36-136}}{2} = \frac{6 \pm \sqrt{-100}}{2}$$

$$= \frac{6 \pm \sqrt{-1}\cdot\sqrt{100}}{2}$$

$$\boxed{\lambda = 3 \pm 5i}$$