Week 14 Monday §6.1-6.4

Orthogonal sets
Orthogonal projection
Inner product (dot product)
Gram-Schmidt Process (G-S)

ANW 9
Daytona June 26th
airdate on
NBC.

Defn: $v \circ w = \sum v_i * w_i$

"$\circ$" dot product
or innerproduct
"mu~~lti~~ply vectors"

So if $v, w \in \mathbb{R}^3$

$$V \circ W = V_1 W_1 + V_2 W_2 + V_3 W_3$$

input 2-vectors
output 1 NUMBER.

Ex.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \circ \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = -1 + 0 + 12 = \underline{\underline{11}}.$$

Ex.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 1 + 0 - 1 = \underline{\underline{0}}.$$

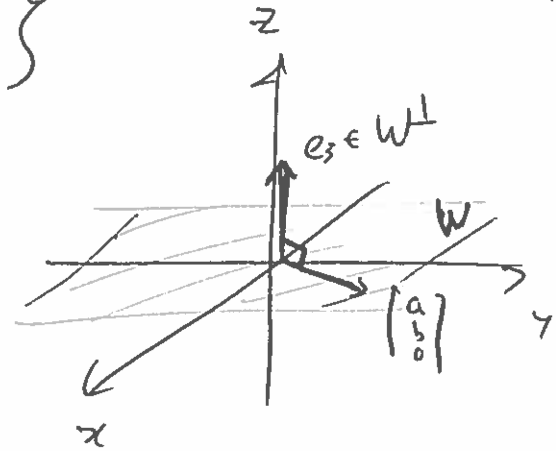
These are orthogonal (means the same thing as perpendicular)

Defn: (Orthogonal complement)
If $W \subseteq \mathbb{R}^n$ is a subspace Then
$W^\perp$ = orthogonal complement of W

$$W^\perp \overset{defn}{=} \{ x \in \mathbb{R}^n \mid x \circ w = 0 \text{ for all } w \in W \}$$

EX. $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

What is $W^\perp$



$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W^\perp$ b/c

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \cdot a + 0 \cdot b + 1 \cdot 0 = 0 \quad \checkmark$$

$W^\perp = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\qquad \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = 0$. too.

note $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^{3\,\perp}$

"perp" means "orthogonal complement of..."

EX. $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$W^\perp = \left\{ \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} ; a, b \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

FACT $\dim W + \dim W^\perp = n$

3

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & [illegible] \\ 0 & 1 & -[illegible] \end{bmatrix} \quad v = s\begin{bmatrix} -[illegible] \\ [illegible] \\ [illegible] \end{bmatrix}.$$

$$W^\perp = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

__Ex.__ $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$

Find a basis for $W^\perp$.

Find $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ s.t. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0 \rightarrow a+b = 0$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0 \rightarrow 2a+b+c = 0$

2 eqns
3 vars

__Why?__ $c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = w \in W$ is __any__ vector in W.

then

$$v \cdot w = v \cdot \left(c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) = c_1 * \left(v \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) + c_2 * \left(v \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0$$

NOTE:
You can just work with the __basis__ of W
don't have to work with general vectors.

$$v \cdot (c * w) = c * (v \cdot w)$$

and

$$v \cdot (w + z) = v \cdot w + v \cdot z$$

"$\cdot$" distributes over vector + and scalar *

FACT If $W = \text{span}\{v_1, v_2\}$

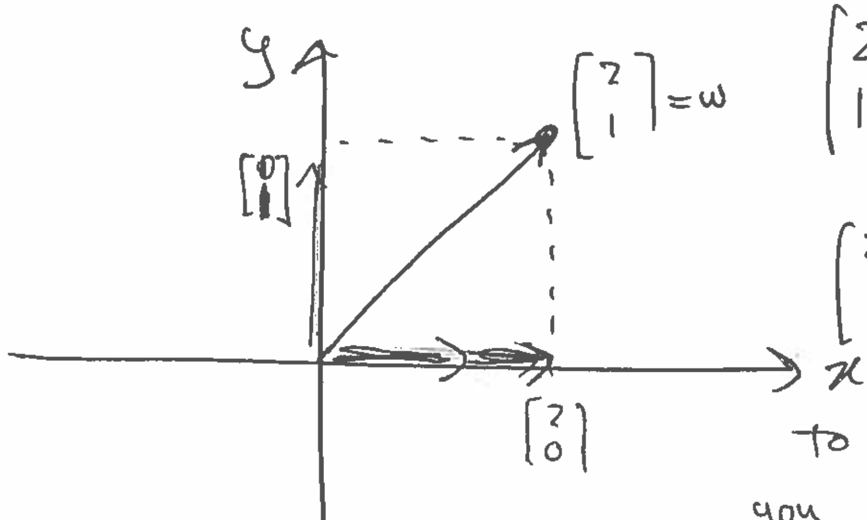
Then
$$W^\perp = \text{null}\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$
$$= \text{null}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$$

That is. We need

$$v_1 \circ x = 0 \quad \longleftrightarrow \quad v_1^T \cdot x = 0$$
$$v_2 \circ x = 0 \quad \longleftrightarrow \quad v_2^T \cdot x = 0$$

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \cdot x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Orthogonal projection



Notice

need 2 e_i

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \cancel{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}\ 2 \qquad 2 \cdot e_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

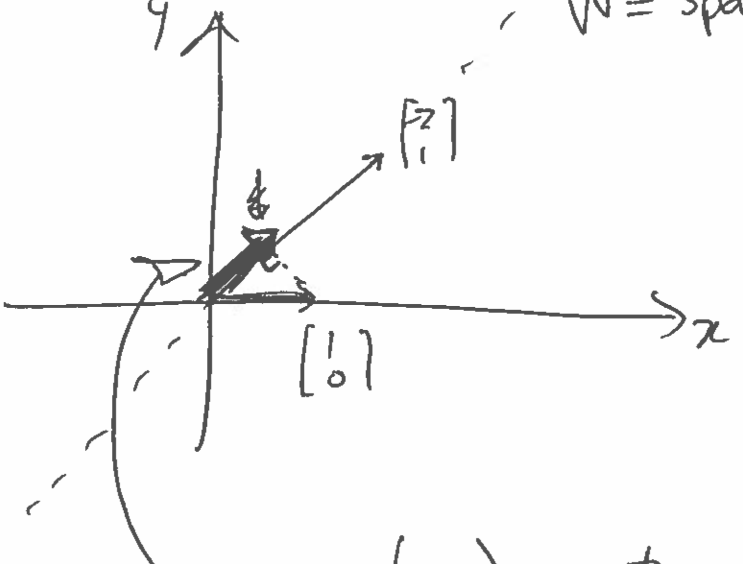
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}\ 1 \qquad 1 \cdot e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

to project w to the x-axis, you can dot w w/ a unit vector in the x-direction

vector w/ length 1. $\longrightarrow$

What about projecting to some other vector?

$W = \operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$

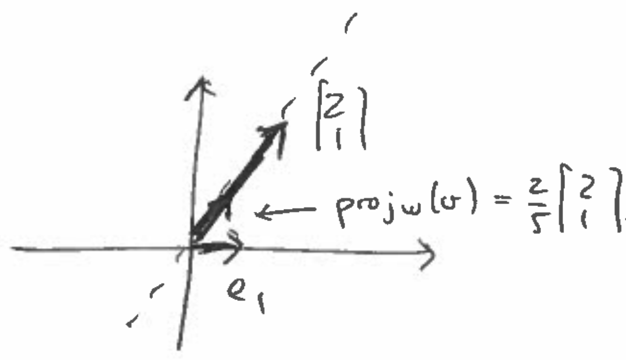


I want to project $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ onto W.

$\operatorname{proj}_W(e_1)$ = the projection onto the subspace W of the vector e_1

(project e_1 onto W)

$$\frac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix}}{\begin{bmatrix}2\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix}}\begin{bmatrix}2\\1\end{bmatrix} = \frac{2}{5}\begin{bmatrix}2\\1\end{bmatrix}$$



Formula:

$$\operatorname{Proj}_w(v) = \frac{v\cdot w}{w\cdot w}\, w$$

$$\begin{bmatrix}2\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix} = 2^2+1^2 = \left\|\begin{bmatrix}2\\1\end{bmatrix}\right\|^2$$

If w_1, w_2, w_3 is a basis for $\mathbb{R}^3$

and $v \in \mathbb{R}^3$.

If we want to write

$$v = c_1 w_1 + c_2 w_2 + c_3 w_3$$

write v in the basis $\{w_1, w_2, w_3\}$ of $\mathbb{R}^3$.

ONE OPTION : Solve $\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = v$

row reduce $\left[\begin{array}{ccc|c} w_1 & w_2 & w_3 & v \end{array}\right]$

NEW OPTION USING "$\circ$" (if w_1, w_2, w_3 are orthogonal only!!) : Use "$\circ$" to find the components c_1, c_2, c_3

$$v = \underbrace{\frac{v \circ w_1}{w_1 \circ w_1}}_{c_1} \cdot w_1 + \underbrace{\frac{v \circ w_2}{w_2 \circ w_2}}_{c_2} w_2 + \underbrace{\frac{v \circ w_3}{w_3 \circ w_3}}_{c_3} w_3.$$

The components of v with respect to $\{w_1, w_2, w_3\}$

Ex. Write $\begin{bmatrix}2\\1\\1\end{bmatrix}$ as a linear comb.

of $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

in order to use the formula these must be ORTHOGONAL

Soln.

$$\begin{bmatrix}2\\1\\1\end{bmatrix} = c_1\begin{bmatrix}2\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

$$c_1 = \frac{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\0\\1\end{bmatrix}}{\begin{bmatrix}2\\0\\1\end{bmatrix}\cdot\begin{bmatrix}2\\0\\1\end{bmatrix}} = \frac{5}{5} = 1 \qquad c_2 = \frac{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}}{\begin{bmatrix}0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}} = \frac{2}{2} = 1 \qquad c_3 = \frac{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}} = \frac{1}{1} = 1$$

Note

$$\begin{bmatrix}2\\1\\1\end{bmatrix} \neq \begin{bmatrix}2\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Ex. write $\begin{bmatrix}2\\1\\1\end{bmatrix}$ as a lin. comb of $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

these are ORTHOGONAL

$$\begin{bmatrix}2\\1\\1\end{bmatrix} = \frac{5}{5}\begin{bmatrix}2\\0\\1\end{bmatrix} + \frac{0}{5}\begin{bmatrix}-1\\0\\2\end{bmatrix} + \frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix} + 0\begin{bmatrix}-1\\0\\2\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\1\\1\end{bmatrix}.$$

✓

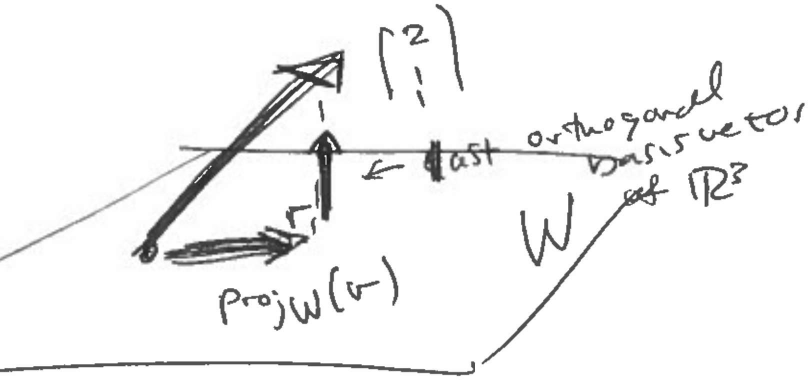
How to project v to $W = \operatorname{span}\{w_1, w_2\}$.

<u>Step 1</u>: Make sure w_1, w_2 are <u>orthogonal</u>.

(otherwise the formula won't work.)

Project $v = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto $W = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

$$\operatorname{proj}_W(v) = \frac{v \cdot w_1}{w_1 \cdot w_1} w_1 + \frac{v \cdot w_2}{w_2 \cdot w_2} w_2$$



$$\frac{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}{\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Ans.

$$\boxed{\operatorname{proj}_W\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.}$$

Week 14 wed.

Continue ORTHOGONALITY

Recall: The dot product "$\circ$" can be used to test whether two vectors v, w are orthogonal.

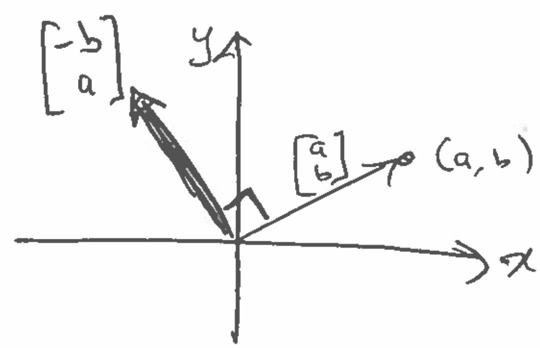
$$V \circ W = v_1 w_1 + v_2 w_2 + v_2 w_3 = 0 \iff v, w \text{ are orthogonal.}$$

NOTE: $\underset{\text{dot}}{V \circ W} = \underset{\text{matrix mult.}}{V^T * W}$

EX. Find a vector orthogonal to $v = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$.

ans $w = \begin{bmatrix} -b \\ a \end{bmatrix}$

Check



$$V \circ W = \begin{bmatrix} a \\ b \end{bmatrix} \circ \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} = (a)(-b) + (b)(a)$$

$$= -ba + ba = 0 \quad \checkmark$$

Q: Are there other vectors that are orthogonal to $v = \begin{bmatrix} a \\ b \end{bmatrix}$?

Ans. any scalar of w will be perp (orthogonal) to v

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} -cb \\ ca \end{bmatrix} = a(-cb) + b(ca)$$
$$= c(-ab + ba)$$
$$= c \cdot 0 = 0.$$

IN FACT. If $W \subseteq \mathbb{R}^n$ (subspace) and we define $W^\perp = \{$all the vectors in $\mathbb{R}^n$ which are orthogonal to every vector in $W\}$

$$= \{x : x \cdot w = 0 \ \forall w \in W\}$$

If W is a subspace

Then

$W^\perp$ is a subspace too.

Subspaces have **bases** and **dimensions**

$W = \mathrm{span}\{v_1, v_2, \dots, v_k\}$

$$W^\perp = \mathrm{null}\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{bmatrix}$$

$W = \mathrm{col}(A)$ "easy to remember formula"

$\mathrm{col}(A)^\perp = \mathrm{null}(A^T).$

Option 1: naive option.

$$\begin{bmatrix}-1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}c_1+c_2\\2c_2\\c_1+3c_2\end{bmatrix} = -(c_1+c_2) - 2c_2 + c_1+3c_2$$

$$= -c_1 - 3c_2 + c_1 + 3c_2$$

$$= 0 \checkmark$$

Option 2: use properties of dot product

1) $v \cdot (w+z) = v \cdot w + v \cdot z$

2) $v \cdot (cw) = c(v \cdot w)$

$$\begin{bmatrix}-1\\-1\\1\end{bmatrix} \cdot \left(c_1 \begin{bmatrix}1\\0\\1\end{bmatrix} + c_2 \begin{bmatrix}1\\2\\3\end{bmatrix}\right)$$

$$= c_1 \underbrace{\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\end{bmatrix}\right)}_{0} + c_2 \underbrace{\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix}\right)}_{0}$$

to see it $v \in W^\perp$

$= 0 + 0$

$= 0 \checkmark$

In the future, we can can just check that

$v \cdot w_1 = 0$
$v \cdot w_2 = 0$

w_1, w_2 are a basis for W.

basis for W

$W = \text{span}\{w_1, w_2, w_3\}$ comprised of ORTHOGONAL vectors. 5

Formula for orthogonal projection

$$\text{proj}_W(v) = \left(\frac{v \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{v \cdot w_2}{w_2 \cdot w_2}\right) w_2 + \left(\frac{v \cdot w_3}{w_3 \cdot w_3}\right) w_3$$

scalar "component of v along w_1"

.... along w_2

NOTE: $\text{proj}_W(v)$ is a linear combination of $\underbrace{w_1, w_2, w_3}$

basis for W

So $\text{proj}_W(v)$ belongs to W.

WARNING the formula only works if

$$\left.\begin{aligned} w_1 \cdot w_2 &= 0 \\ w_2 \cdot w_3 &= 0 \\ w_1 \cdot w_3 &= 0 \end{aligned}\right\} \text{ orthogonal basis for } W.$$

So when you apply the formula you have to make sure that w_1, w_2, w_3 are ORTHOGONAL.

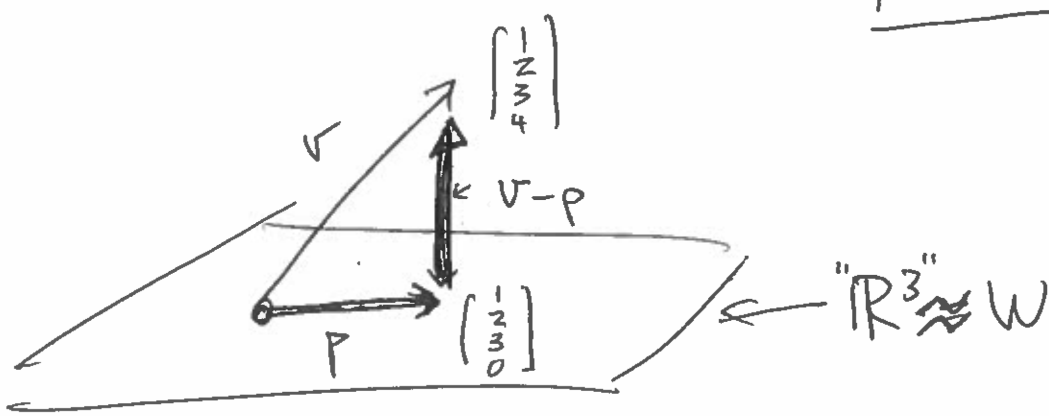
Ex. $v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $W = \text{span}\{e_1, e_2, e_3\} \subseteq \mathbb{R}^4.$

Q: $\text{Proj}_W(v) = ?$

Soln. $$\text{proj}_W(v) = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot e_1}{e_1 \cdot e_1} \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot e_2}{e_2 \cdot e_2} \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot e_3}{e_3 \cdot e_3} \overset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$= \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}}.$$



Note $(v-p) \cdot p = 0$ So $v - p \in W^{\perp}$

Ex. Find an orthogonal basis for

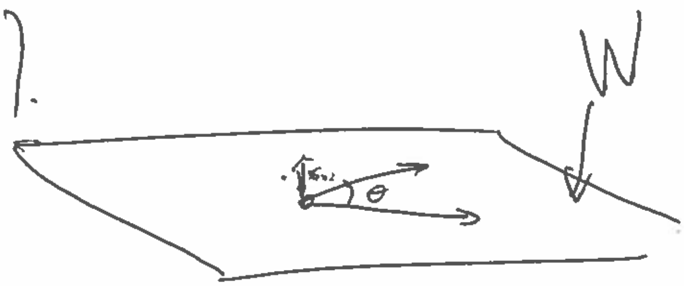
$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$\begin{array}{cc} \uparrow & \uparrow \\ w & v \end{array}$$

Step 1: compute

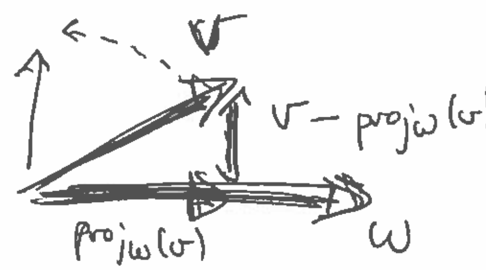
$$\text{proj}_w(v) = \frac{v \cdot w}{w \cdot w}\, w$$

$$= \frac{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$



$$= \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ 4/3 \\ 0 \\ 4/3 \end{bmatrix}.$$



Step 2: Compute $\hat{v}$ "v hat"

$$\hat{v} = v - \text{proj}_w(v)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 4/3 \\ 0 \\ 4/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 0 \\ -1/3 \end{bmatrix} \xrightarrow{\text{scale up}} \underset{\text{new}}{\hat{v}} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

<u>Check that</u>

$$W = \text{Span}\left\{ \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\1\end{bmatrix} \right\} \overset{?}{=} \text{Span}\left\{ \underbrace{\begin{bmatrix}-1\\2\\0\\-1\end{bmatrix}}_{\hat{v}}, \underbrace{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}_{w} \right\}$$

and that $\hat{v} \cdot w = 0$

$$\begin{bmatrix}-1\\2\\0\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix} = (-1) + 2 + 0 - 1$$
$$= -2 + 2 = 0 \checkmark$$

<u>Note</u> $\begin{bmatrix}-1\\2\\0\\-1\end{bmatrix} \in W$ since

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-4\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\\0\\1\end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix}-1\\2\\0\\-1\end{bmatrix} \in W.$$

The G-S process

↑ Gram-Schmidt

will output orthogonal basis if given input any basis for subspace W.

META

input $\{w_1, w_2, w_3\}$ basis for subspace W.

Step 1: Find $\hat{w}_2$.

$$\hat{w}_2 = w_2 - \frac{w_2 \circ w_1}{w_1 \circ w_1} w_1$$

Step 2: Find $\hat{w}_3$

$$\hat{w}_3 = w_3 - \frac{w_3 \circ w_1}{w_1 \circ w_1} w_1 - \frac{w_3 \circ \hat{w}_2}{\hat{w}_2 \cdot \hat{w}_2} \hat{w}_2.$$

EX. Find an orthogonal basis for $W = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{bmatrix}1\\ \text{[illegible]}\\0\end{bmatrix}\right.$

next time...