Least squares

Week 15
monday

Formula

$$\hat{X} = (A^T A)^{-1} * A^T b$$

Ex Find the least squares solution to the system

$$\begin{cases} 4x + 0y = 2 \\ 0x + 2y = 0 \\ x + y = 11 \end{cases}$$

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Step 1: compute $(A^T \cdot A)^{-1}$

$$A^T \cdot A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \frac{1}{17 \cdot 5 - 1} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Step 2: Compute $A^T \cdot b$ $\quad \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$

Step 3: Compute

$$\hat{X} = (A^T \cdot A)^{-1} \cdot A^T b$$

$$= \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$= \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So $x = 1$, $y = 2$ does the best job possible when trying to solve the system

$$\begin{cases} 4x + 0y = 2 \\ 0x + 2y = 0 \\ x + y = 11 \end{cases}$$

~~Ex. $W = \text{Span}\left\{\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$, $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.~~

~~Compute $\text{Proj}_b(W)$~~

Ex. Find the least squares soln.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

$$\hat{x} = (A^T \cdot A)^{-1} \cdot A^T b.$$

$$(A^T \cdot A) = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \frac{1}{132-121} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix}$$

$$A^T \cdot b = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}.$$

$$\hat{x} = (A^T A)^{-1} A^T b = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} -88 + 121 \\ -44 + 66 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 33 \\ 22 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

$x = 3, y = 2$

~~[illegible]~~

If $W \subseteq \mathbb{R}^4$ is a line in $\mathbb{R}^4$.

Then what is the dim of $W^\perp$?

Ans. For example, if

$$W = \text{Span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\right\}$$

$$W^\perp = \left\{x \in \mathbb{R}^4 : x \cdot \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = 0\right\}$$

$$= \left\{x \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 + 4x_4 = 0\right\}$$

$W^\perp = \text{null}\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ has dim 3.

In general $\boxed{\dim W + \dim W^\perp = n}$