# Worksheet 3, Math 1553

Sections from Lay $5^{th}$ edition: 1.4, 1.5

## Exercises

1. If possible, give an example of a vector $\vec{b}$, and a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ (where $k$ can be 2, 3, etc.) such that $\vec{b}$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, but there are at least two ways to write the linear combination.

2. If possible, give an example of a $2 \times 3$ matrix $A$ that is in reduced echelon form, has two pivot columns, and satisfies
$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3. Identify the solution of the system and write it in parametric vector form. Give a geometric description of the solution set.
$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

4. Rewrite the matrix equation below as a vector equation:
$$\begin{pmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$
What is the minimum number of the matrix columns needed to express $\begin{pmatrix} 8 \\ -1 \end{pmatrix}$ as a linear combination of them? Write such linear combination.

5. For the following situations, determine (a) whether the equation $A\vec{x} = \vec{0}$ has a nontrivial solution and (b) whether the equation $A\vec{x} = \vec{b}$ has at least one solution for every possible $\vec{b}$ in $\mathbb{R}^m$, and explain:
   (i) $A$ is a $3 \times 3$ matrix with 3 pivots.
   (ii) $A$ is a $3 \times 3$ matrix with 2 pivots.
   (iii) $A$ is a $3 \times 2$ matrix with 2 pivots.
   (iv) $A$ is a $2 \times 4$ matrix with 2 pivots.

6. Indicate whether the statement true or false. If it is true, in one or two sentences, explain why. If false, give a counter example or explain why in one or two sentences.
   a) A non-trivial solution $\vec{x}$ to $A\vec{x} = \vec{0}$ has all non-zero entries.
   b) Any $3 \times 2$ matrix $A$ with two pivot columns has a non-trivial solution to $A\vec{x} = \vec{0}$.

# Worksheet 3 Answers, Math 1553

Sections from Lay $5^{th}$ edition: 1.4, 1.5

1. $\vec{b} = (2,1)$, and $\vec{v}_1 = (4,2)$, $\vec{v}_2 = (-2,-1)$. Then $\vec{b} = 0\vec{v}_1 - \vec{v}_2 = \vec{v}_1 + \vec{v}_2$.
2. $$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$
3. We first write the augmented matrix for this system. It is

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}.$$

We can row reduce by first replacing row 2 with itself plus 4 times row 1, then replace row 3 with itself plus row 2. At this point we are in echelon form, with

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that there is no pivot in the rightmost column, so the system is consistent, and there is also no pivot in the column corresponding to $x_3$, so it is a free variable. To put in reduced echelon form, we scale row 2 by dividing by 3, then replace row 1 with itself minus 3 times row 2. This gives

$$\begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, the solutions are $x_1 = -2 + 5x_3$, $x_2 = 1 - 2x_3$, $x_3$ free. In parametric vector form, we would have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

Geometrically, the solution is represented by a line in $\mathbb{R}^3$ going through the point $(-2,1,0)$ and parallel to the vector $[5 \ \ -2 \ \ 1]$.

4. **Solution** The matrix equation:

$$\begin{pmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

is expressed as a vector equation as

$$-3\begin{pmatrix} -3 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 8 \end{pmatrix} + 4\begin{pmatrix} -4 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 9 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

At least two columns are needed to express $\begin{pmatrix} 8 \\ -1 \end{pmatrix}$ as a linear combination of them. For instance

$$7\begin{pmatrix} -4 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 9 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

5. (i) $A$ is a $3 \times 3$ matrix with 3 pivots. **Solution**: (a) $A$ has a pivot in every column, so there are no nontrivial solutions since there are no free variables. (b) $A$ has a pivot in every row, so there is at least one solution for every $\vec{b}$ in $\mathbb{R}^3$ - in fact, there is exactly one solution for every $\vec{b}$, since there are no free variables.

   (ii) $A$ is a $3 \times 3$ matrix with 2 pivots. **Solution**: (a) $A$ does not have a pivot in every column, so there are nontrivial solutions since there are free variables. (b) $A$ does not have a pivot in every row, so there is not a solution for each $\vec{b}$ in $\mathbb{R}^3$.

   (iii) $A$ is a $3 \times 2$ matrix with 2 pivots. **Solution**: (a) $A$ has a pivot in every column, so there are no nontrivial solutions since there are no free variables. (b) $A$ does not have a pivot in every row, so there is not a solution for each $\vec{b}$ in $\mathbb{R}^3$.

   (iv) $A$ is a $2 \times 4$ matrix with 2 pivots. **Solution**: (a) $A$ does not have a pivot in every column, so there are nontrivial solutions since there are free variables. (b) $A$ has a pivot in every row, so there is at least one solution for every $\vec{b}$ in $\mathbb{R}^3$ - in fact, there are infinite solutions for every $\vec{b}$, since there are free variables.

6. (a) False. Counterexample:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ has the solution } \vec{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \quad x_2 \in \mathbb{R}$$

   (b) False. There are no free variables, so non-trivial solutions do not exist.