Worksheet 4, Math 1553

Sections from Lay 5^{th} edition: 1.7, 1.8, 1.9

Exercises

1. The columns of a 7×5 matrix are linearly independent. How many pivot columns does it have?
2. List all possible echelon forms of the matrix in each case (indicate non zero elements with $*$ and pivots with a square).
   (a) A is a 2×2 matrix with linearly dependent columns
   (b) A is a 4×2 matrix $[\vec{v}_1\ \vec{v}_2]$, and $\vec{v}_2$ is not a multiple of $\vec{v}_1$.
   (c) A is a 4×3 matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, such that the set $\{\vec{v}_1\ \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ is not in Span$\{\vec{v}_1\ \vec{v}_2\}$.
3. Consider the vectors
   $$\vec{v}_1 = \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$
   Note that the sum of $\vec{v}_1$ and twice $\vec{v}_2$ is equal to $\vec{v}_3$. Denote A as the matrix $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$. Answer the following questions, with justification.
   (a) Is the set of vectors $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ linearly independent?
   (b) Without row reducing, does A have a pivot in every column?
   (c) Does the linear transform $T(\vec{x}) = A\vec{x}$ map $\mathbb{R}^3$ onto $\mathbb{R}^3$? Is the linear transform one-to-one?
4. The linear transform $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points across the horizontal x_1 axis and then reflects them across the line $x_2 = x_1$. Construct the standard matrix representation of T.
5. Show that T is a linear transformation by constructing a matrix that implements the mapping.
   (a) $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$
   (b) $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4$

Answers to Worksheet 4

1. A 7×5 matrix must have exactly five pivot columns for the columns of the matrix to be linearly independent. Otherwise, with fewer than five pivots, then there is at least one column that can be represented as a linear combination of the other columns.

2. Possible echelon forms:

   (a) A is a 2×2 matrix with linearly dependent columns:

   $$\begin{bmatrix} \square & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \square \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   (b) A is a 4x2 matrix $[\vec{v}_1\ \vec{v}_2]$, and $\vec{v}_2$ is not a multiple of $\vec{v}_1$.

   $$\begin{bmatrix} \square & * \\ 0 & \square \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \square \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   (c) A is a 4x3 matrix $[\vec{v}_1\ \ \vec{v}_2\ \ \vec{v}_3]$, such that $\{\vec{v}_1\ \ \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ is not in Span$\{\vec{v}_1\ \vec{v}_2\}$.

   $$\begin{bmatrix} \square & * & * \\ 0 & \square & * \\ 0 & 0 & \square \\ 0 & 0 & 0 \end{bmatrix}$$

3. Consider the vectors

   $$\vec{v}_1 = \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

   Note that the sum of $\vec{v}_1$ and twice $\vec{v}_2$ is equal to $\vec{v}_3$. Denote A as the matrix $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$. Answer the following questions, with justification:

   (a) Is the set of vectors $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ linearly independent? **Solution**: No. Since $\vec{v}_3$ is a linear combination of the vectors $\vec{v}_1$ and $\vec{v}_2$ – $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$ specifically – the set is linearly dependent.

   (b) Without row reducing, does A have a pivot in every column? **Solution**: No. Since the vectors are linearly dependent, the matrix A with those vectors as its columns must have free variables, and therefore does not have a pivot in every column.

   (c) Does the linear transform $T(\vec{x}) = A\vec{x}$ map $\mathbb{R}^3$ onto $\mathbb{R}^3$? Is this linear transform one-to-one? **Solution**: No to both. Since the matrix A does not have a pivot in every column, it also does not have one in every row, as it is a square matrix. Since there is not a pivot in every row, there will be some vectors $\vec{b}$ in $\mathbb{R}^3$ whereby $A\vec{x} = \vec{b}$ will not have a solution; hence, not every vector in the co-domain is part of the range of the transform, so the transform is not "onto". The lack of a pivot in every column means there are free variables, so that even for those $\vec{b}$ that are in the range of T, there are infinite solutions $\vec{x}$ to achieve those $\vec{b}$, so the transform is not one-to-one.

4. Let the linear transform $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ first reflect points across the horizontal x_1 axis and then reflect them across the line $x_2 = x_1$. Find the standard matrix representation of T.

   Solution: In order to find the standard matrix of a linear transform, we need only determine how it acts on the columns of the identity matrix I_n. Here we have $n = 2$, so we need to consider $\vec{e}_1 = [1\ 0]$ and $\vec{e}_2 = [0\ 1]$. Take $\vec{e}_1$. T first reflects this across the x_1 axis, but since the vector is already on that axis, nothing happens. Then, the point is reflected across the diagonal line $x_2 = x_1$, which will move $[1\ 0]$ to point $[0\ 1]$. So, $T(\vec{e}_1) = [0\ 1]$. Now take $\vec{e}_2$. T will first reflect across the x_1 axis, turning $\vec{e}_2$ into $[0\ -1]$. Then T will reflect across the $x_2 = x_1$ line, putting the point at $[-1\ 0]$. So, $T(\vec{e}_2) = [-1\ 0]$. The matrix of T is $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$, giving

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

5. The matrix that implements T:

   (a) $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

   $$A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   (b) $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4$

   $$A = \begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix}$$