Worksheet 9, Math 1553

Sections from Lay 5^{th} edition: 5.3, 5.5

Exercises on 5.3

1. If possible, construct matrices P and D such that the matrix

$$A = \begin{pmatrix} 6 & 3 \\ -4 & -1 \end{pmatrix}$$

can be diagonalized as $A = PDP^{-1}$, where D is diagonal. You do not need to compute P^{-1}.

2. If possible, give an example of a square matrix, A, that has the following properties.
   (a) A is 2×2, is in echelon form, invertible, and cannot be diagonalized.
   (b) A is 2×2, is in echelon form, singular, and can be diagonalized.
   (c) A is 3×3, is in echelon form, singular, and can be diagonalized.

3. Fill in the blanks to express the diagonalization of A and B:

$$A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 3 & \quad & \quad \\ 0 & & \\ 2 & & \end{bmatrix} \begin{bmatrix} \quad & 0 & 0 \\ 0 & \quad & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \quad & \quad & -3 \\ & 0 & 6 \\ & & 8 \end{bmatrix}^{-1}.$$

$$B = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 1 & 3 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & \quad & \quad \\ 0 & & \\ \quad & & \end{bmatrix} \begin{bmatrix} \quad & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \quad \end{bmatrix} \begin{bmatrix} -3 & \quad & \quad \\ 0 & & \\ \quad & & \end{bmatrix}^{-1}.$$

Exercises on 5.5

(I) Answer the following short questions. Justify your reasoning.

   (a) If matrix A is 2×2 and is not invertible, can any of the eigenvalues of A have a non-zero imaginary component?
   (b) If A is a real matrix with complex eigenvalue $\lambda = a + ib$ (where $b \neq 0$), can the eigenvector $\vec{v}$ associated with λ be purely imaginary; that is, can $\vec{v} = i\vec{x}$, with $\vec{x}$ in $\mathbb{R}^n$?
   (c) If 2×2 real matrix A has purely imaginary eigenvalues ib and $-ib$ (and assume $b > 0$), by what angle does the similar rotation/scaling matrix $C = P^{-1}AP$ rotate vectors, and by what factor does it scale them? Be sure to answer for both of the eigenvalues.

(II) Matrix A is a composition of a rotation and a scaling. Give the angle of rotation, ϕ, and the scale factor, r.

$$A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$$

(III) Let $A = \begin{pmatrix} 4 & -1 \\ 2 & 6 \end{pmatrix}$. Construct an invertible matrix P and a matrix C from the complex eigenvalues of A, such that $A = PCP^{-1}$.