Additional Review Set for Midterm 2

This is a set of questions that students can use to help them prepare for an upcoming exam. Note that the topics and sections from the textbook that the midterms covers are listed in the syllabus.

Topics Covered in this Review Set

This review set focuses on the following sections from Lay 5^{th} Edition:

- 2.2, 2.3, 2.4, 2.5, 2.6, 2.8, 2.9, 3.1, 3.2, 3.3, 4.9, 5.1, 5.2

Review Set Solutions

A primary goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, *solutions are not are provided for the additional review problem sets*. This is intentional: most upper level courses do not have recitations, MML, and solutions for everything. Students must develop and implement their own strategies to check their solutions in those courses.

In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses, and beyond.

Recommended Strategies for Preparing for Midterms

Students are encouraged to prepare for their exams by:

1. completing all of the problems in the practice review sets,
2. completing additional problems from the textbook,
3. spend most of their preparing by solving problems and solving lots of problems,
4. studying with other people and comparing study strategies with them,
5. asking questions during office hours and/or piazza,
6. start studying as early possible,
7. spread your studying out over many days,
8. identify your goals (i.e. - what you want to accomplish by taking this course), write them down, and align your study strategies with them,
9. review the learning objectives of the course and align your study strategies with them,

10. organize your studying: schedule times you want to study in an online calendar or in your planner.

There are *many* other effective study strategies and resources that you can take advantage of at Georgia Tech. Take time to think about what works for you to reach your goals. And ask other students, your instructors, and TAs what approaches they recommend for preparing for exams.

The MML Study Plan

The MML Study Plan has hundreds of problems you can solve that are automatically graded for you. MML will tell you if your work is correct and offers a few different study aids. To access the study plan for a specific textbook section:

1. navigate to mymathlab.com and log in
2. select your course
3. select Lay Linear Algebra (the online textbook)
4. select a chapter
5. select a section
6. click study plan

Contents

1 True/False Exercises

Indicate whether each statement is true or false.

1.1) If A and B are invertible $n \times n$ matrices, then the sum $A + B$ is also invertible.

1.2) The set of all possible solutions to $A\vec{x} = \vec{0}$ is a subspace.

1.3) The set of all possible solutions to $A\vec{x} = \vec{b}$ is a subspace.

1.4) If A^n is invertible for some integer n, then A is also invertible.

1.5) Any three vectors in $\mathbb{R}^2$ will form a basis for $\mathbb{R}^2$.

1.6) If U is an echelon form of matrix A, then rank(U) = rank(A).

1.7) If the nullspace of a square matrix contains only the zero vector, then the matrix is invertible.

1.8) Any four linearly independent vectors in $\mathbb{R}^4$ forms a basis for $\mathbb{R}^4$.

1.9) The rank of an invertible $n \times n$ matrix is always n.

1.10) If A is invertible, then the columns of A^{-1} span $\mathbb{R}^n$.

1.11) A 4×6 matrix could have rank as large as 6.

1.12) $H = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 = 0\}$ is a subspace.

1.13) $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 - x_2 = 0, x_4 = 1\}$ is a subspace.

1.14) $S = \{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 + 2x_2 = 0\}$ is a subspace.

1.15) $L = \{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 \leq 0\}$ is a subspace.

1.16) $P = \{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$ is a subspace.

1.17) An $n \times n$ matrix with a zero row must be singular.

1.18) All elementary matrices are triangular.

1.19) If matrix A is 4×3, then the set of solutions to $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^3$.

1.20) If matrix A is row reduced to echelon form to produce matrix E, then the null spaces of A and E are the same.

1.21) If matrix A is row reduced to echelon form to produce matrix E, then the column spaces of A and E are the same.

1.22) If $\vec{u}$ and $\vec{v}$ are in subspace S, then $\vec{u} + \vec{v}$ is also in S.

1.23) If $\vec{u}$ is in subspace S, then any vector in Span($\vec{u}$) is also in S.

1.24) The column space of a matrix is a subspace, pivotal columns are vectors, and the rank of a matrix is an integer.

1.25) Swapping the columns of matrix A does not change the value of det(A).

1.26) Swapping the rows of matrix A does not change the value of det(A).

1.27) The rank of matrix A and the dimension of Col(A) are always equal to each other.

1.28) Any matrix can be reduced to echelon form by multiplying the matrix by a set of elementary matrices.

1.29) If a matrix is in upper triangular form then it is also in echelon form.

1.30) If a matrix is in echelon form then it is also in upper triangular form.

1.31) If a matrix is in upper triangular form then all of the elements on the main diagonal must be non-zero.

1.32) Suppose that E_1 and E_2 are any 2×2 elementary matrices. Then $E_1E_2 = E_2E_1$.

1.33) If A is square, and $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then det(A) $= 0$.

1.34) If A, B, and C are square, A is invertible, and $A(B - C)$ is equal to a zero matrix, then $B = C$.

1.35) If A, B, and C are square, A is invertible, and $A(B - A^TC^T)$ is equal to a zero matrix, then $B = (AC)^T$.

1.36) Every elementary matrix is invertible.

1.37) If A is invertible, then $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$.

1.38) The product of invertible matrices is also invertible.

1.39) Every matrix can be expressed as a product of elementary matrices.

1.40) If A is a 3×3 matrix and the equation $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has a unique solution then A is invertible.

1.41) If B, C, D are square matrices and $BC = BD$, then $C = D$.

1.42) If B, C, D are square invertible matrices and $BC = BD$, then $C = D$.

1.43) The transpose of an elementary matrix is an elementary matrix.

1.44) If A has N columns then rank$A$ + dim(Nul(A)) = N.

1.45) The set of all probability vectors in $\mathbb{R}^n$ forms a subspace of $\mathbb{R}^n$.

1.46) The set of eigenvectors of an $n \times n$ matrix, that are associated with an eigenvalue, λ, span a subspace of $\mathbb{R}^n$.

1.47) An eigenspace is a subspace spanned by a single eigenvector.

1.48) If $\lambda \in \mathbb{R}$ is a non-zero eigenvalue of A with corresponding eigenvector $\vec{v}$, then $A\vec{v}$ is parallel to $\vec{v}$.

1.49) If A is $n \times n$ and A has n distinct eigenvalues, then the eigenvectors of A span $\mathbb{R}^n$.

1.50) An eigenvalue of a matrix could be associated with two linearly independent eigenvectors.

1.51) Row operations on a matrix do not change its eigenvalues.

1.52) If A is singular, then non-zero vectors in the null space of A are also eigenvectors of A.

1.53) An example of a 2×2 matrix, that only has eigenvalue zero, is the 2×2 zero matrix.

1.54) A steady-state vector for a stochastic matrix is an eigenvector.

1.55) If v_1 and v_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

1.56) A number c is an eigenvalue of A if and only if the system $(A - cI)x = 0$ has a nontrivial solution.

1.57) If $(\lambda - r)^k$ is a root of the characteristic polynomial of A, then r is an eigenvalue of A with geometric multiplicity k.

1.58) If A is a square matrix, $\vec{v}$ and $\vec{w}$ are eigenvectors of A, then $\vec{v} + \vec{w}$ is also an eigenvector of A.

1.59) An eigenvalue of a matrix could be associated with two linearly independent eigenvectors.

Note that students are welcome (and encouraged) to post their answers to these questions on Piazza to discuss them. Likewise with the remaining problems on this worksheet.

2 Example Construction Exercises

2.1) **If possible**, write down an example of a quantity (or quantities) with the following properties. If it is not possible to do so, write *not possible*. You do not need to justify your reasoning.

(a) A 4×3 matrix C, that is in RREF, and satisfies dim(Null(C)) = 2.

(b) A matrix C so that $CA = I_2$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(c) A 2×2 matrix, A, that is singular, but has a solution to $A\vec{x} = \vec{b}$ for all possible $\vec{b} \in \mathbb{R}^2$.

(d) A 3×3 lower triangular matrix whose determinant is equal to 2.

(e) A 3×4 matrix in RREF that has a two dimensional null space.

(f) A matrix that is not invertible and has an LU factorization.

(g) A 3×3 elementary matrix that swaps the first and third columns of another matrix.

(h) A is a matrix with three rows. Construct the elementary matrix E so that EA is the matrix obtained by adding 3 times the first row of A with the second row of A.

(i) A 2×2 elementary matrix, E, such that $\det(E) = 0$.

(j) A 2×2 elementary matrix, E, such that $T(\vec{x}) = E\vec{x}$ performs a reflection.

(k) A 3×2 matrix with column space given by vectors in $\mathbb{R}^3$ such that $x_1+2x_2+3x_3 = 0$.

$$\begin{pmatrix} -1 & \_\_ \\ 0 & -1 \\ \_\_ & \_\_ \end{pmatrix}$$

(l) A 3×3 elementary matrix that is also stochastic.

(m) A 2×2 matrix that is the standard matrix for a one-to-one transform, and whose eigenvalues are 1 and 0.

(n) A 4×4 matrix, A, whose column space is the susbpace

$$\{\vec{x} \in \mathbb{R}^4 \,|\, x_1 + x_2 + x_3 + x_4 = 0\}$$

(o) A 3×3 upper triangular matrix, all entries either 0, -1 or 1, and whose column space is the plane $x_1 + x_2 + x_3 = 0$.

(p) A 3×3 matrix with one pivot and whose null space is the plane $x_1+2x_2+3x_3 = 0$.

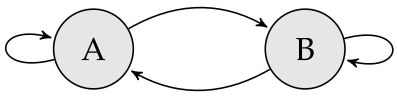
(q) A 2×2 stochastic matrix, A, that is in echelon form. A steady-state vector for the Markov chain $\vec{x}_{k+1} = A\vec{x}_k$ is $\vec{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(r) A 5×5 matrix that has one eigenvalue that has algebraic multiplicity 5 and geometric multiplicity 2.

(s) Matrices A and B that are similar but have different characteristic equations.

(t) Matrices A and B that are not similar but have the same eigenvalues.

(u) A two state Markov chain that is not regular.



2.2) If possible, give an example of a 3×4 matrix A, in row reduced echelon form, with the properties below.

(a) rank(A) = 3

(b) rank(A^T) = 4

(c) rank(A) = 4

(d) dim(Null(A)) = 2

(e) dim(Null(A)) = 2, and rank(A) = 1

2.3) If possible, give an example of a matrix,A, with the following properties.

(a) A 2×2 matrix whose column space is the line $3x_1 + x_2 = 0$, and whose null space is the line $5x_1 + x_2 = 0$.

(b) A is a 2×2 matrix whose column space is S.

$$S = \{\vec{x} \in \mathbb{R}^2 \,|\, x_1 + x_2 = 1\}$$

Hint: is the column space of a matrix a subspace? Is S a subspace? What is a subspace?

(c) A is 3×2, Null$A$ is the line $x_1 = kx_2$, where $k \in \mathbb{R}$. A basis for the column space of A is the vector

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

(d) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. The dimension of the nullspace of A is zero.

(e) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and dim(Null($A$)) = 1.

(f) $T_A : \mathbb{R}^3 \to \mathbb{R}^3$. A basis for the range of T is the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(g) The nullspace and column space of A are the same. A is a 2×2 matrix.

2.4) We may construct a basis for Col(A) using the pivot columns of A. Do we always have to use the pivot columns to form a basis for Col(A)? If not, give an example of any non-zero matrix whose columns are spanned by vectors that are *not* pivot columns.

3 Multiple Choice, Short Answer, and Fill in the Blank Exercises

3.1) The characteristic polynomial of a matrix A is $(\lambda-3)(\lambda-1)^2(\lambda+4)^3$.

(a) The characteristic equation is ______________________.

(b) det______________ $= (\lambda-3)(\lambda-1)^2(\lambda+4)^3$.

(c) The dimensions of A are ______________.

(d) Matrix A has ______________ distinct eigenvalues.

(e) The eigenvalues of A are equal to ______________.

(f) The eigenvectors of A have ______________ elements.

(g) The vectors in the null space of A have exactly ______________ elements.

(h) The dimension of the null space of A is equal to ______________.

(i) The rank of A is equal to ______________.

(j) Is A invertible? (yes or no)

(k) Is $\det(A) = 0$? (yes or no) ______________.

3.2) The eigenvalues of A can be determined by inspection. What are they?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Hints: one eigenvalue has a geometric multiplicity of two, and the sum of the geometric multiplicities is at most the number of columns of the matrix. The geometric multiplicity of an eigenvalue is the dimension of the corresponding eigenspace.

3.3) By inspection, determine whether A is invertible.

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 0 & 0 \\ 8 & 5 & 1 \end{pmatrix}$$

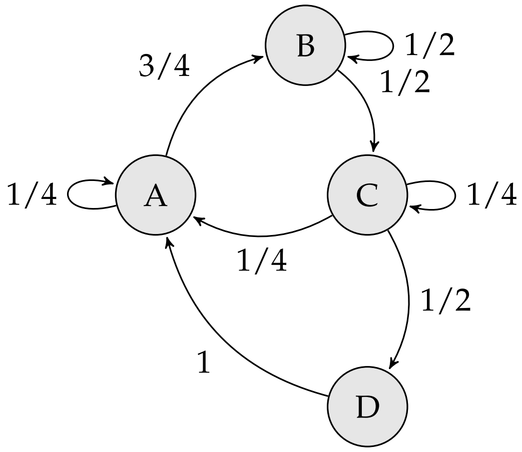
3.4) Fill in the blanks.

(a) A is a 5×4 matrix in RREF whose rank is equal to 3. How many different matrices can you construct that meet these criteria? ___________

(b) For what values of k does the matrix have two distinct real eigenvalues? ___________

$$\begin{pmatrix} -7 & k \\ 2 & -4 \end{pmatrix}$$

3.5) Identify the transition matrix for the Markov chain below.



3.6) Indicate which of the following situations are possible. You do not need to explain your reasoning. If the situation is possible give an example.

(a) Matrix A has dimensions 5×7, rank(A) = 2, dim(Nul(A)) = 3.

(b) Matrix A has dimensions 5×7, rank(A) = 5, dim(Nul(A)) = 2.

(c) Matrix A has dimensions 5×7, rank(A) = 6, dim(Nul(A)) = 1.

(d) Matrix A is $n \times n$, A is singular, det(A) = 0, and $T(\vec{x}) = A\vec{x}$ is not one-to-one.

(e) E is a singular elementary matrix.

(f) Matrix A is a standard matrix for a linear transform that performs a projection in $\mathbb{R}^n$, and $\dim(\text{Null}(A)) = 1$.

(g) Matrix A is a standard matrix for a linear transform that performs a counterclockwise rotation in $\mathbb{R}^n$, $\dim(\text{Null}(A)) = 0$, $\det(A) = 1$.

(h) V is a set of linearly independent vectors in $\mathbb{R}^3$ that form a basis for the column space of an invertible 3×3 matrix.

(i) V is a set of linearly independent vectors in $\mathbb{R}^3$ that form a basis for the null space of an invertible 3×3 matrix.

3.7) Suppose A is an $n \times n$ invertible matrix. Fill in the blanks.

(a) The system $A\vec{x} = \vec{b}$ always has a ______________ solution.

(b) The columns of A form a ______________ set.

(c) The columns of A ______________ $\mathbb{R}^n$.

(d) The only solution to $A\vec{x} = \vec{0}$ is ______________ .

(e) The transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$, is ______________ and ______________.

(f) The null space of A is ______________.

(g) The dimension of the column space of A is ______________.

3.8) Suppose A is an 11×3 matrix and that $T(x) = Ax$. If T is one-to-one, the dimension of the null space of A is ______________.

3.9) Which of the following, if any, are subspaces of $\mathbb{R}^3$?

(a) $\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_1 = a, x_2 = a - 1, x_3 = a + 2, a \in \mathbb{R}\}$

(b) $\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_1 + x_2 + x_3 = 0\}$

(c) $\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_1 + x_2 + x_3 = 4\}$

(d) $\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_1 < x_2 < x_3\}$

(e) $\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_2 = x_3 = 0\}$

3.10) Two $m \times n$ matrices A and B have the same columns, but in different orders. For each of the quantities below, list whether they are the same (S) for both A and B, regardless of how the columns are ordered, or different (D).

(a) **S** **D:** The echelon form.
(b) **S** **D:** The number of free variables.
(c) **S** **D:** The rank.
(d) **S** **D:** The basis for the Null Space.
(e) **S** **D:** The free variables.
(f) **S** **D:** The Null space.
(g) **S** **D:** The dimension of the Column Space.
(h) **S** **D:** The Column Space.
(i) **S** **D:** The Reduced Echelon Form.
(j) **S** **D:** The pivotal columns.
(k) **S** **D:** The basis for Column space.
(l) **S** **D:** The Reduced Echelon form.
(m) **S** **D:** The Nullity (the dimension of the null space).

3.11) Consider the LU factorization:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(a) If k can be any real number, will the product LU always yield the same matrix A, regardless of the value of k?

(b) Given your answer to the previous question, is the LU factorization of a matrix necessarily unique?

4 Computation Exercises

4.1) Without any computation, state the value of the determinant of C.

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

4.2) Construct a basis for the subspace

$$D = \{\vec{x} \in \mathbb{R}^5 \mid x_1 + x_2 = x_5,\ x_3 = 0\}.$$

4.3) For what values of h and k are the matrices singular? You can assume that h and k must be real numbers.

(a) $A = \begin{pmatrix} 5 & 1 \\ 0 & 0 \end{pmatrix} - hI_2$

(b) $B = \begin{pmatrix} 0 & 1 & h \\ -3 & 10 & 0 \\ -3 & 5 & k \end{pmatrix}$

(c) $C = \begin{pmatrix} 1 & 1 \\ k & k^2 \end{pmatrix}$

4.4) For what values of k, if any, do the vectors not form a basis for $\mathbb{R}^4$?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ k \end{bmatrix}.$$

4.5) The determinant $\begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 5$. Determine the value of the determinants below.

(a) $\begin{vmatrix} 4 & 5 & 6 \\ a & b & c \\ 1 & 2 & 3 \end{vmatrix}$

(b) $\begin{vmatrix} 4 & 6 & 5 \\ a & c & b \\ 1 & 3 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 2a & 2b & 2c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$

4.6) Compute the determinant of C.

$$C = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 7 & 1 \end{pmatrix}$$

4.7) Use a determinant to determine whether the vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}$$

4.8) An economy consists of three sectors, X, Y, and Z. For every 100 units that sector X produces, X consumes 10%, Y consumes 20%, and Z consumes 30%. For every 100 units that sector Y produces, X consumes 20%, Y consumes 30%, and Z consumes 10%. For every 100 units that sector Z produces, X consumes 30%, Y consumes 10%, and Z consumes 20%. A final demand for units X, Y, and Z that these sectors produce is specified by the vector

$$\vec{d} = \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$$

Construct a linear system, that when solved, would give the production level, $\vec{x}$, that would satisfy this demand. You do not need to solve the linear system.

4.9) Let $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix}$ be a 4×4 matrix whose determinant is equal to 2. Compute the value of the determinant $\begin{vmatrix} \vec{d} & \vec{b} & 3\vec{c} & \vec{a} \end{vmatrix}$.

4.10) Let A, B and C be 4×4 matrices with $\det A = 2, \det B = -3, \det C = 5$. Determine the value of the determinants of the following matrices.

$$AB, \quad AC^{-1}B, \quad B^TC^2, \quad A^3B^{-1}C^T, \quad 4C,$$

4.11) Compute the area of the parallelogram determined by $\mathbf{0}, \vec{u}, \vec{v}, \vec{u} + \vec{v}$ where

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

4.12) R is the parallelogram determined by $\vec{p}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and $\vec{p}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. If $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, what is the area of the image of R under the map $\vec{x} \mapsto A\vec{x}$?

4.13) The vector $\vec{x} = (1, 12, 3)^T$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, where $\vec{v}_1 = (1, 0, -1)^T$ and $\vec{v}_2 = (-1, 3, 2)^T$. Compute $[\vec{x}]_{\mathcal{B}}$.

4.14) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transformation that first rotates vectors in $\mathbb{R}^2$ counterclockwise by θ radians about the origin, then reflects them about the line $x_1 = x_2$. By inspection, what is the value of the determinant of A? You should compute its value to check your answer.

4.15) Consider the sequence of row operations that reduce matrix A to the identity.

$$A = \underbrace{\begin{pmatrix} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{E_2 E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_3 E_2 E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_4 E_3 E_2 E_1 A} = I_3$$

(a) Construct the four elementary matrices E_1, E_2, E_3, and E_4.

(b) Consider the matrix products listed below. Which (if any) represents A, and which (if any) represents A^{-1}?

i. $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$

ii. $E_4E_3E_2E_1$

iii. $E_1E_2E_3E_4$

iv. $E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$

4.16) Suppose $A_n \in \mathbb{R}^{n\times n}$ and $a_{ij} = \min(i, j)$. Compute the value of $\det(A_n)$.

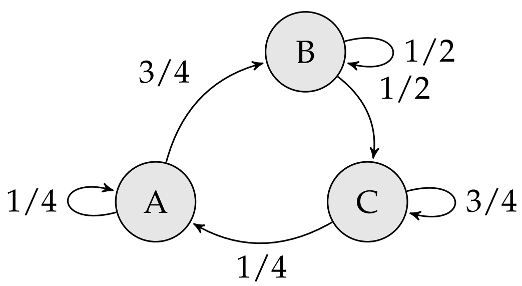
4.17) For what values of k does the matrix have two distinct real eigenvalues?

$$\begin{pmatrix} -7 & k \\ 2 & -4 \end{pmatrix}$$

(a) B is a 2×2 matrix with two identical rows. What is one of the eigenvalues of B equal to?

(b) Suppose again that B is a 2×2 matrix with two identical rows. Identify a relationship between the two entries in each row such that the second eigenvalue of B is 1.

4.18) Identify the transition matrix and the steady state vector for the Markov chain below.



4.19) Consider the Markov chain

$$\vec{x}_k = A\vec{x}_{k-1} = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix} \vec{x}_{k-1}, \quad k = 1, 2, 3, \ldots, \quad \vec{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are 1 and 0.6. Analyze the long-term behavior of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

4.20) Consider the dynamical system $\vec{x}_k = A\vec{x}_{k-1},\ k = 1, 2, 3, \ldots$, where

$$A = \begin{pmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvalues of A are 1 and $\frac{1}{4}$. Analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

4.21) If P is a regular stochastic matrix with steady state vector $\vec{w} = \frac{1}{10}\begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$ and $\vec{x}_0 = \frac{1}{10}\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$, what does the sequence $\vec{x}_k = P^k\vec{x}_0$ converge to?

4.22) Suppose $\vec{v}_1$, $\vec{v}_2$ are eigenvectors of an 3×3 matrix A that correspond to eigenvalues λ_1 and λ_2.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = \frac{1}{4}$$

Vector $\vec{p}$ is such that $\vec{p} = 4\vec{v}_1 - 5\vec{v}_2$. Calculate the values of $A\vec{p}$, $A^2\vec{p}$, and $A^k\vec{p}$. What does $A^k\vec{p}$ tend to as $k \to \infty$?

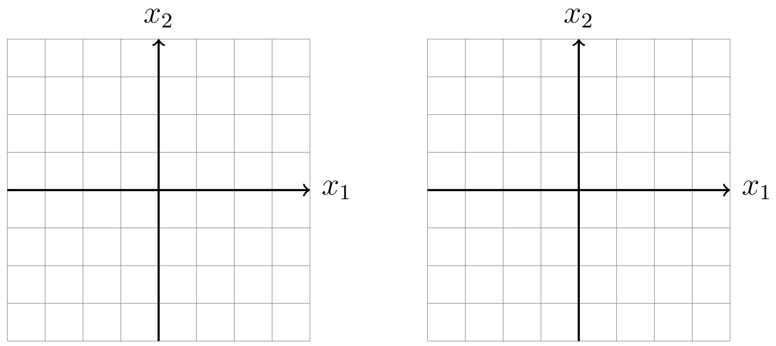
5 Sketching Questions

5.1) Sketch the nullspace and column space of $A = \begin{pmatrix} 1 & -2 \\ 4 & -8 \end{pmatrix}$.

5.2) T is the linear transform $x \to Ax$ that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the nullspace of A, the range of the transform, and the column space of A. How are the range and column space related to each other?

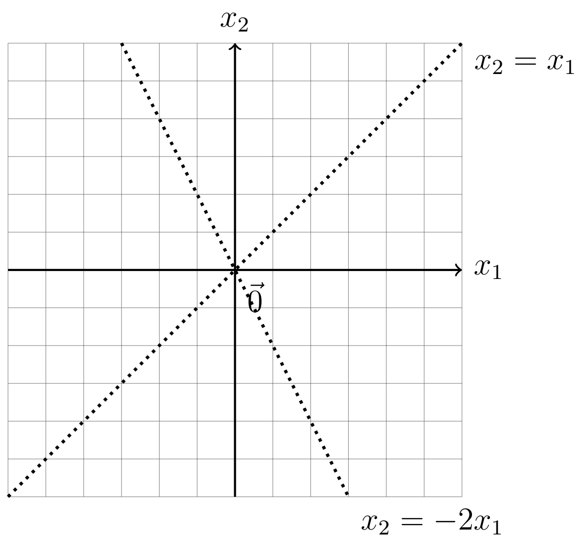
5.3) On the grids below, sketch a) the null space of $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$, and b) the eigenspaces of A.

(a) Null(A) (b) eigenspaces

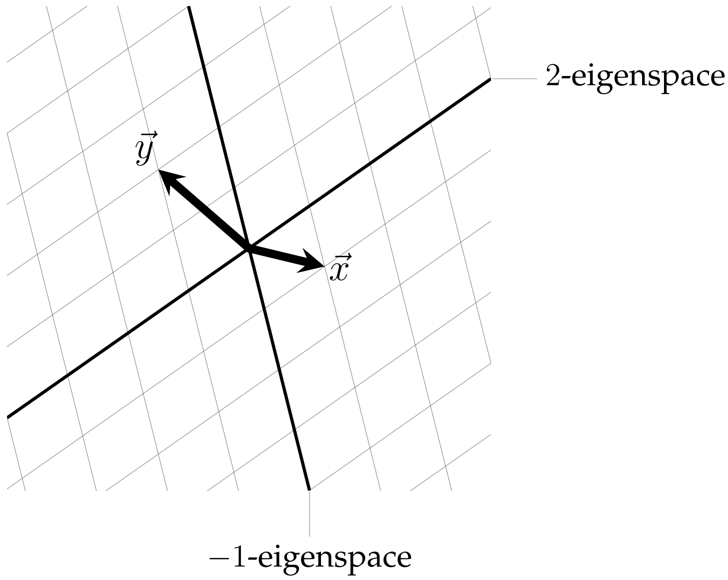


5.4) $T_A = A\vec{x}$ is a linear transformation that stretches vectors along the line $x_2 = -2x_1$ by a factor of 2. Vectors along the line $x_2 = x_1$ are scaled by a factor of $\frac{1}{2}$ under T_A. The two lines are shown below. Sketch the images of the following vectors under transformation T_A on the grid below. Do not determine the elements of A.

$$\vec{p} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \vec{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \vec{r} = 2\vec{p} - \vec{q} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$



5.5) A 2×2 matrix A has eigenvalues -1 and 2, with eigenspaces indicated in the picture. Draw $A\vec{x}$ and $A\vec{y}$.



6 ISyE Exercises

The following set of questions **are definitely within the scope of what Math 1554 students are expected to be able to do**. A few years ago, conversations between School of Math faculty and Industrial and Systems Engineering (ISyE) faculty over led to the development of these questions: they emphasize concepts we explore in Math 1554 that are also needed in various ISyE courses that use Math 1554 as a pre-requisite.

6.A) Which of the following matrices (if any) are non-singular? If the matrix is non-singular, compute its inverse.

(a) $X = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

(b) $Y = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

6.B) Suppose $B \in \mathbb{R}^{2\times 2}$, $C \in \mathbb{R}^{2\times 2}$, A is the block matrix $(B\ C)$. Column vectors $\vec{u}$ and $\vec{v}$ have two elements, and $\vec{x} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}$

(a) Circle the following operations that are defined and determine the dimensions of the result.

$$B\vec{u} \qquad A\vec{x} \qquad \vec{v}^T A\vec{x}$$

(b) True or false: $A\vec{x} = B\vec{u} + C\vec{v}$

6.C) Let S be the set of all possible linear combinations of vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

(a) Is the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in S?

(b) True or false: the sum of any two vectors in S is also in S.

(c) True or false: the difference of any two vectors in S is also in S.

(d) True or false: any scalar multiple of a vector in S is also in S.

(e) True of false: S is a subspace.