

• •			Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
			1	1/8 - 1/12	1.1	WS1.1	1.2	W51.2	1.3
1.7	LINEAR INDEP	PENDENCE	2	1/15 - 1/19	Break	WS1.3	1.4	WS1.4	1.5
		The homogeneous equations in Section 1.5 can be studied from a different r	3	1/22 - 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
		by writing them as vectors equations. In this way, the focus shifts from the solutions of $A\mathbf{x} = 0$ to the vectors that appear in the vector equations.	4	1/29 - 2/2	1.9,2.1	WS1.9,2.1	Exam 1, Review	Cancelled	2.2
								• •	• •
• •				0 0				• •	• •
• •		at which is the track of a tend of a track of a structure to the			ON	IG OMG O	MG OMG O	MG OM	G
•	DEFINITION	An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be <b>linearly independent</b> if the vector equation	ndent	E			000		2
		$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p = 0$			·	10.00		2XX	
		has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be <b>linearly dependent</b> if there exist weights $c_1, \ldots, c_p$ , not all zero, such that	ndent	t .			* YTW		
		$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_p\mathbf{v}_p=0$	(2)	) -				P	
						IT'S FIN	ALLY FRID	DAY II	
• •				• •			A STATE OF STATE OF STATE	make	ameme or
				• •				• •	• •
• •	· · · ·	· · · · · · · · · · · · · · · · · · ·						• •	•
	Equivalent defin	* Ax=0 has only the trivial sum FACTS!	12	A ·	is mxn	A = [v,	. Vn] • •		• •

•	Equivalent defn.	* A× * A	(=0 has	has only A=[1 a pive	y the t 11 uz u of in o	,ivial ( 1p] 1evry (	sulu cdi	•	•	•	•	FA	CT	*	t 11	N.	A 7m	is The	w V	n×v1 {v	A ,u,3	ј = [ ј и	N1	. Vn.	]	•	•	•	•	•	•
				A=[v		]				•				¥	16	ζ٧	,,\	lyp9 G	re l	in in	1 11	er V	NYI	η.				•	•		
	• • • •			• •	• •									*	Ax:	= 0	has	; G	£10	22 V	oK										
																		=7	30	h,,V	n f I	in 1	æl.								
	· r			( a) ( ) (	c value	( 06	e lin	indi	lin	906																					
	. EX. Which a	f the tol	lowing	5615 01	t anoibi	5 044									•													•	•		
6	6 517	ra1	r o'	17		ſ	- ,	2	0	ດ໌	1																		•		
0	3 0			ζ			0	1	1	Ð	1	$\vee$									•			•				•			
	([[],	ردו	LI	J			1	3	١	0	J		•	•				•			•		•		•	•					
				• •																											
				• •																	•			•							
		• •		• •	• •					•											•		•	•				•	•		
			•	• •	• •			•			•				•						•		•					•	•		
		7 (-	277	7				1	7	· 0	2																				
	$(3)$ $2 _2$	2	4	(			.	7	4	0		~	,																		
	( L)	1,1	- 0	J				1	Z	C	>				•								•					•			
							Ļ								•						•		•	•	•	•		•	•		
											•														•	•					
-			-	-		-	-	-	-	-		-	-	-	-	-	-	-	-	-				-	-	-	-	-			-

	1.5								
DEFINITION	An indexed set of vectors $\{v_1, \ldots, if$ the vector equation	$\mathbf{v}_p$ in $\mathbb{R}^n$ is said to be <b>linear</b>	ly independent	Week Date	s Lecture	Studio	Lecture	Studio	Lectu
	$x_1 \mathbf{v}_1 + x_2$	$\mathbf{v}_2\mathbf{v}_2+\cdots+\mathbf{x}_p\mathbf{v}_p=0$		1 1/8	- 1/12 1.1	WS1.1	1.2	WS1.2	1.3
	has only the trivial solution. The s	et $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be <b>line</b> tot all zero, such that	arly dependent	2 1/15	- 1/19 Break	WS1.3	1.4	WS1.4	1.5
	$c_1 \mathbf{v}_1 + c_2$	$_{2}\mathbf{v}_{2}+\cdots+c_{p}\mathbf{v}_{p}=0$	(2)	3 1/22	- 1/26 1.7	WS1.5,1.7	1.8	WS1.8	1.9
				4 1/29	- 2/2 1.9,2.1	W51.9,2.1	Exam 1, Review	Cancelled	2.2
					• • • •				
Equivalent	t defin. * Ax=0 has only the	trivial sub	FACTS:	It A	is mxn	A=[v, v	n]		
	A = [v, vz	.vp]		2 M 7 M	then Symme	uriz lin dee			
	* A has a pivot in	every cd.	v 10	- <u>5</u> .	1 Jaco lin ind	ther m3 n.			
	H=[V,,	د م.	+ 14	: 199777	optore			• • •	
 r .		ace line indivitive dep	×β	1×=0 ho	s a free vo	r 7 lin dee.		• • •	
<u>Ex</u> . Whi	ich of the tollowing sets of vecto.	S DALE IN MY			2 301,,	/	0 0	• • •	•
6 / F.	- CO1 CO17	(1,2,n;2)						• • •	•
0 3 6			v						
([;	ן, נטן, נוי ג	1310]							
					• • • •			• • •	
· (2) 5	$\begin{vmatrix} 1 \\ 2 \end{vmatrix} \begin{vmatrix} 2 \\ 4 \end{vmatrix} $								
. [	$\lfloor 1 \rfloor, \lfloor 2 \cup S$		~						
								• • •	
									•
• • • •									
								• • •	

2	$\left\{ \begin{bmatrix} 1\\0\\1 \end{bmatrix}, \begin{bmatrix} 2\\2\\0 \end{bmatrix}, \begin{bmatrix} 0\\7\\7 \end{bmatrix} \right\}$	· · · · · · · · · · · · · · · · · · ·	•	· · ·	<ul> <li>.</li> <li>.&lt;</li></ul>	· · · · · · · · · · · · · · · · · · ·	<ul> <li>.</li> <li>.&lt;</li></ul>	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·		· · ·	•	· · ·
	$\begin{cases} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}, \end{cases}$			· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·		<ul> <li>.</li> <li>.&lt;</li></ul>						· · · · · · · · · · · · · · · · · · ·
.         .         .           .         .         .           .         .         .           .         .         .           .         .         .           .         .         .           .         .         .           .         .         .           .         .         .           .         .         .           .         .         .		· · · · · · · · · · · · · · · · · · ·	· · · ·	· · · · · · · · · · · · · · · · · · ·	<ul> <li>.</li> <li>.</li></ul>	<ul> <li>.</li> <li>.</li></ul>	<ul> <li>.</li> <li>.&lt;</li></ul>	· · · · · · · · · · · · · · · · · · ·	· · · · · ·		· · · · · · · · · · · · · · · · · · ·		· · · · · · · · · · · · · · · · · · ·
			•	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·			· · · · · · · · · · · · · · · · · · ·	•	· · · · · · · · · · · · · · · · · · ·
   		· · ·	•	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·		· · ·	•	· · · · · · · · · · · · · · · · · · ·
· · · · ·	· · · · · · · · · · · · · · · ·	• •	•	• •	• •	• •	· ·	• •	• •	• •	• •	•	• •

a multip vectors i	two vec le of the s a mult	other tiple o	<b>v</b> <sub>1</sub> , <b>v</b> <sub>2</sub> . The f the	2} is set othe	linea is lin er.	arly early	depe y ind	nder	nt if nden	at le t if a	east of	ne of nly i	f the y	her o	ors of th	is ne		•	•	•	•	•	•	•	• •
				• •			• •				• •		• •				• •								
				• •							• •		• •												
		• •		• •			• •				• •		• •				• •								
	• •	• •		• •						•	• •		• •		•	•	• •						•		
		·									• •		• •		•	•	• •						•		
THEOD			Char	anto	rizot	lon	of L	Inco		long	andor	+ 6	**												•
THEOR			Arei	acte	ada			ned		Jehe	ftm	it St	15	last	0.00	ia 11		<b>b</b> 1	-	a da		F			
			and o	only	if at l	least	$= \{ v \\ one \}$	of the	· · · V	p; 0	rs in 9	or n	lines	vect	mb	is II inat	ion (	iy d of th	e pe	her	s Ir				
			fact,	if S	is lir	nearl	ly de	pend	lent	and	$\mathbf{v}_1 \neq$	0, th	nen so	ome	Vi	(wit	h j	> 1	) is	a li	nea	r			
		1.0	com	oinat	ion c	of the	e pre	cedi	ng v	ecto	ors, v <sub>1</sub>	,	$, \mathbf{v}_{j-}$	1.	,										
				• •									• •				• •								
		• •		• •			• •						• •				• •								
		• •		• •			• •			•	• •		• •				• •						•		
							• •				• •		• •			•									
0 0 0	• •		0		0				0		• •		• •				• •			•	•		•	0	
	• •	• •	•		•		• •	•	•	•	• •	•	• •	•	•	•	• •		•	•	•	•	•	•	
	• •	• •	•	• •	set is	auto	matic.	More	over,	1 nec	orem 8	w111 D	е а кеу	resu		worn		tter c	napte	TS.	•	•	•	•	• •
	   	    	DREM	M 8	set is If is p	auto a set linea > n.	matic. conta rly de	More ins m	over, ore vo	I nec	s than t	will b here a t $\{\mathbf{v}_1,$	е а кеу are enti	ries in ]	It for n eac $\mathbb{R}^n$ is	worn th vec line	c in ia	tter c	napte the se dent	rs.	•	•	•	•	· ·
	   	THE(	DREM	M 8	set is If is <i>p</i>	a set linea $> n$ .	matic. conta rly de	more ins m pende	over, ore v	i nec	s than t	will b here $t \{v_1, v_1\}$	e a key are entr	resu ries i .} in l	n eac $\mathbb{R}^n$ is	worn th ve	c in ia	tter c	napte the se dent	et if		•	· · ·	•	· · ·
		THE(	DREM	M 8	Set IS If is p	a set linea $> n$ .	conta rly de	ins m pende	over, ore v	i nec	s than t	wiii D here : t { <b>v</b> <sub>1</sub> ,	e a key are entr	ries in ]	It for n eac $\mathbb{R}^n$ is	worn h vec line	c in ia	then then	napte the se dent	et if		•		•	· · ·
	· · · · · · · · · · · · · · · · · · ·	THE(	DREM	M 8	set is If is <i>p</i>	a set linear $> n$ .	conta rly de	more ins m pende	over,	i nec	s than t	wiii d here : t { <b>v</b> <sub>1</sub> ,	are entriand $\mathbf{v}_p$	ries i } in }	It for n eac $\mathbb{R}^n$ is	worn h veo line	c in ia	tter c	napte the so	rs.		• • • • • • •			
n []		    	DREM	VI 8	If is p	a set linea > n.	conta rly de	More .	over, ore v	i nec	s than to	will be the set of th	аге entti	ries i } in ]	It for n eac $\mathbb{R}^n$ is	worn h vea line	c in ia	then epen	the so	et if	· · · · · · · · · · · · · · · · · · ·	• • • • • • • •			
	· · · · · · · · · · · · · · · · · · ·	THE(		VI 8	set is If is <i>p</i>	a set linea > n.	conta	ins m pende	ore v	ectors inat is,	s than t		ате enti , <b>v</b> <sub>p</sub>	ries i } in ]	n eac ℝ" is	worn h ve	ctor, 1 arly d	then epen	the sudent	et if	· · · · · · · · · · · · · · · · · · ·	• • • • • • • • •			
		THE(		M 8	set is If <i>p</i>	a set linea > n.	conta	ins m pende	ore v nt. Tł	ectory nat is:	s than to so that the solution of the solution		are entri	ries i in l	n eac R <sup>n</sup> is	worn h vec line:	c in iz	then epen	the solution	et if	· · · · · · · · · · · · · · · · · · ·				
		THE * * * * * * * * * * * * * * * * *		VI 8	set is If is p	a set linea > n.	conta rly de	More ins m pende	over,	I nectornatis,	s than the standard sta		are entries $\mathbf{v}_p$	ries i } in }	n eac R <sup>n</sup> is	worn h vee l line:	c in ia c in ia arly d	then epen	the standard	et if					
		THEC *** *** *** *** *** *** *** *		N 8	set is If <i>p</i>	a set linea > n.	contat rly de	more ins m pende	ore v nt. Th	ectors nat is.	s than 1		are entri	ries i in ) in )	n eac	worn h vec line	ctor, t arly d	then epen	the so	rs.					
		THEC *** ***         		M 8	set is If <i>p</i>	a set linea	conta rly de	ins m pende	ore v nt. Th	I nec	s than 1	will D here : t {v <sub>1</sub> ,	are entre en	ries i in l	n eac R <sup>n</sup> is	worn h vec line:	c in ia	then epen	the solution	et if					
				M 8	set is If is <i>p</i>	a set linea > n.	conta	ins m pende	ore v nt. Th	ectors nat is.	s than 1 , any sc		are entit , ψ <sub>p</sub>	ries i i in i i i i i i i i i i i i i i i i	n eac R" is	worn h vee lines	ctor, t ctor, t	then epen	the subscription of the su	et if					
					set is If is p	a set linea > n.	contat	ins m pende	ore v ont. Th	ectors	s than the second secon		are entities of the second sec	ries i i n 1 } in 1	n eac R <sup>n</sup> is	worn h vei line	c in iz	then epen	the set						
		THE(			set is p	a set linea > n.	conta rly de	ins m pende	ore v	ectors nat is,	s than 1		are entries of a second	resu ries i } in ]	n eac	worn h vee line:	c in iz	inter c	the sudent						
		THEC *** THEC *** *** *** *** *** *** *** *			set is	a set linea > n.	conta rly de	ins m pende	ore vont. Th	i nectors is,	s than 1		are entities 	ries i i i n ] } in ]	n eac R <sup>n</sup> is	worn h vee l lines	ctor, t arly d	inter c	the sudent						
		THE THE THE THE THE THE THE THE			set is	a set linea > n.	conta rly de	ins m pende	ore v. nt. Th	i nec	s than 1 , any sc 		are entri- 	ries i } in ]		worn h ves line:	c in iz	then epen	the sed dent	et if					
					set is	a set linea > n.	conta rly de	ins m pende		ectors nat is,	s than 1 , any sc 		re entit 	ries i } in 1		worn	ctor, i arly d	- inter c	napte						

If a set  $S = {\mathbf{v}_1, \dots, \mathbf{v}_p}$  in  $\mathbb{R}^n$  contains the zero vector, then the set is linearly dependent. **THEOREM 9** lin der?  $\begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{cases}$ ÉX.

			+																																
			L	4																															
			5	-	· ·																														
			/			In 1	Exe	rci	ses	11	-14	4, f	ind	the	e va	lue	e(s)	of	h f	or	whi	ich	the	e ve	ecto	rs									
					1	are	lin	ear	ly a	dep	end	len	t. Ju	isti	fy e	eacl	h ai	nsw	er.																
							Г			Г		Г	Г	1				Г	2	٦	Г	6	٦	Г	Г										
						11		1	1		-	2		-1			12		4		1	-0													
								_		,	-;	2	,	5		1	12.		-4	1		2		1											
							L	. 4	+ ]	L			L	n				L	1		L-	-3		L 4	-										
							•	•												•															
					•		•												•						•	•									
					•	•	•	•											•	•					•	•									
					•	•	•	•	•										•	•					•	•									
																										•									
	•	•			•	•	•	•							•				•		•			•	•	•						•			•
					•	•	•	•	•										•	•					•	•									
					•	•	•	•				•			•	•			•		•	•	•	•	•	•						•		•	
							•																		•	•				•					
_			_								,		,				,										-	-	-	c	,	,			,
				-	-	-	-		-	-			-	-		-	-	-	-	-		-	-		-								-	-	
															•									•											
					•														•						•										
					•	•	•		•							•			•		•		•		•	•				•					
					•	•	•	•				•			•	•			•		•	•	•	•	•	•						•		•	

# 1.7 EXERCISES

In Exercises 1-4, determine if the vectors are linearly indepen-0 -8 5 -4 -3 0 3 -7 4 0 -14 dent. Justify each answer. 5. -4 5 1 0 3 2 5 4 6 \_3 2. 1. 0 4 0 5 4 3 0 -3 3 7. 5 1 7 .2 5 0 3. 1 In Exercises 9 and 10, (a) for what values of h is  $v_3$  in

Span  $\{\mathbf{v}_1, \mathbf{v}_2\}$ , and (b) for what values of h is  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  linearly dependent? Justify each answer.

In Exercises 5-8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

### 62 CHAPTER 1 Linear Equations in Linear Algebra

9. 
$$\mathbf{v}_1 = \begin{bmatrix} 1\\ -3\\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3\\ 9\\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5\\ -7\\ h \end{bmatrix}$$
  
10.  $\mathbf{v}_1 = \begin{bmatrix} 1\\ -5\\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2\\ 10\\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2\\ -9\\ h \end{bmatrix}$ 

In Exercises 11–14, find the value(s) of h for which the vectors are linearly dependent. Justify each answer.

12. -5 5 7 h 11. 13. 14.

Determine by inspection whether the vectors in Exercises 15-20 are linearly independent. Justify each answer.

**15.** 
$$\begin{bmatrix} 5\\1 \end{bmatrix}, \begin{bmatrix} 2\\8 \end{bmatrix}, \begin{bmatrix} 1\\3 \end{bmatrix}, \begin{bmatrix} -1\\7 \end{bmatrix}$$
 **16.**  $\begin{bmatrix} 4\\-2\\6 \end{bmatrix}, \begin{bmatrix} 6\\-3\\9 \end{bmatrix}$   
**17.**  $\begin{bmatrix} 3\\5\\-1 \end{bmatrix}, \begin{bmatrix} 0\\0\\0 \end{bmatrix}, \begin{bmatrix} -6\\5\\4 \end{bmatrix}$  **18.**  $\begin{bmatrix} 4\\4 \end{bmatrix}, \begin{bmatrix} -1\\3 \end{bmatrix}, \begin{bmatrix} 2\\5 \end{bmatrix}, \begin{bmatrix} 8\\1 \end{bmatrix}$   
**19.**  $\begin{bmatrix} -8\\12\\-4 \end{bmatrix}, \begin{bmatrix} 2\\-3\\-1 \end{bmatrix}$  **20.**  $\begin{bmatrix} 1\\4\\-7 \end{bmatrix}, \begin{bmatrix} -2\\5\\3 \end{bmatrix}, \begin{bmatrix} 0\\0\\0 \end{bmatrix}$ 

- 24. A is a  $2 \times 2$  matrix with linearly dependent columns.
- **25.** A is a 4  $\times$  2 matrix,  $A = [\mathbf{a}_1 \ \mathbf{a}_2]$ , and  $\mathbf{a}_2$  is not a multiple of a1.

-2

2

- **26.** A is a  $4 \times 3$  matrix,  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ , such that  $\{\mathbf{a}_1, \mathbf{a}_2\}$  is linearly independent and  $\mathbf{a}_3$  is not in Span  $\{\mathbf{a}_1, \mathbf{a}_2\}$ .
- 27. How many pivot columns must a 7 × 5 matrix have if its columns are linearly independent? Why?
- 28. How many pivot columns must a  $5 \times 7$  matrix have if its columns span ℝ5? Why?
- 29. Construct  $3 \times 2$  matrices A and B such that  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution and  $B\mathbf{x} = \mathbf{0}$  has a nontrivial solution.
- 30. a. Fill in the blank in the following statement: "If A is an  $m \times n$  matrix, then the columns of A are linearly independent if and only if A has pivot columns."

b. Explain why the statement in (a) is true.

Exercises 31 and 32 should be solved without performing row operations. [Hint: Write  $A\mathbf{x} = \mathbf{0}$  as a vector equation.]

**31.** Given 
$$A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$
, observe that the third column

is the sum of the first two columns. Find a nontrivial solution of  $A\mathbf{x} = \mathbf{0}$ .

6 3 32. Given A =-7 5 , observe that the first column In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

- **21.** a. The columns of a matrix A are linearly independent if the equation  $A\mathbf{x} = \mathbf{0}$  has the trivial solution.
  - b. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.
  - c. The columns of any 4 × 5 matrix are linearly dependent.
  - d. If x and y are linearly independent, and if {x, y, z} is linearly dependent, then z is in Span {x, y}.
- 22. a. Two vectors are linearly dependent if and only if they lie on a line through the origin.
  - b. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.
  - c. If x and y are linearly independent, and if z is in Span {x, y}, then {x, y, z} is linearly dependent.
  - d. If a set in R<sup>n</sup> is linearly dependent, then the set contains more vectors than there are entries in each vector.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

23. A is a  $3 \times 3$  matrix with linearly independent columns.

plus twice the second column equals the third column. Find a nontrivial solution of  $A\mathbf{x} = \mathbf{0}$ .

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a *counterexample* to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

- **33.** If  $\mathbf{v}_1, \ldots, \mathbf{v}_4$  are in  $\mathbb{R}^4$  and  $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is linearly dependent.
- 34. If  $v_1,\ldots,v_4$  are in  $\mathbb{R}^4$  and  $v_3=0,$  then  $\{v_1,v_2,v_3,v_4\}$  is linearly dependent.
- **35.** If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are in  $\mathbb{R}^4$  and  $\mathbf{v}_2$  is not a scalar multiple of  $\mathbf{v}_1$ , then  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is linearly independent.
- 36. If v<sub>1</sub>,..., v<sub>4</sub> are in R<sup>4</sup> and v<sub>3</sub> is *not* a linear combination of v<sub>1</sub>, v<sub>2</sub>, v<sub>4</sub>, then {v<sub>1</sub>, v<sub>2</sub>, v<sub>3</sub>, v<sub>4</sub>} is linearly independent.
- **37.** If  $\mathbf{v}_1, \ldots, \mathbf{v}_4$  are in  $\mathbb{R}^4$  and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent, then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is also linearly dependent.
- **38.** If  $\mathbf{v}_1, \ldots, \mathbf{v}_4$  are linearly independent vectors in  $\mathbb{R}^4$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is also linearly independent. [*Hint*: Think about  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$ .]

## Section 1.8 : An Introduction to Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

### 1.8 : An Introduction to Linear Transforms

#### Topics

We will cover these topics in this section.

- 1. The definition of a linear transformation.
- The interpretation of matrix multiplication as a linear transformation.

#### Objectives

For the topics covered in this section, students are expected to be able to do the following.

- Construct and interpret linear transformations in R<sup>n</sup> (for example, interpret a linear transform as a projection, or as a shear).
- 2. Characterize linear transforms using the concepts of
  - ▹ existence and uniqueness
  - ▹ domain, co-domain and range

		7:54 AN	-																																		
																				vve	ек D	ates		Lectu	re		Studio			Lectur	e		Stu	Idio		Lectur	5
				Sect	ion 1.8	· An Ir	stroduct	ion to l	linear		1.8 : T	An In	troduct	ion to I	inear T.	ransfor	ms			1	1.	/8 - 1/	12	1.1			WS1.1			1.2			Ws	\$1.2		1.3	
				0000	1011 2.0	Trans	sforms	1011 00 1	emedi		w	Ve will cov 1. The d 2. The in	ver these to efinition of steroretatic	a linear t of mate	is section. ransformat	tion.	a linear			2	1.	/15 - 1	l/19	Break			WS1.3			1.4			W	51.4		1.5	
					Ch.	apter 1 : L fath 1554 I	inear Equati Linear Algeb	ons ra				transf	ormation.							3	1	/22 - 1	L/26	1.7			WS1.5	1.7		1.8			W	51.8		1.9	
											Fr de	or the top o the folk	ics covered owing.	d in this se	ction, stu	dents are o	expected t	o be able t	o	4	1.	/29 - 2	2/2	1.9,2	1		WS1.9	2.1		Exam 1	L, Revie	ew	Ca	ncelled		2.2	
												<ol> <li>Constr interpr</li> <li>Charas</li> </ol>	ruct and in ret a linear cterize line	terpret lin transform ar transfo	ear transfo as a proje ms using t	ection, or the concep	in R <sup>n</sup> (for as a shear) pts of	example, ).																			
	•											≻ ei ≻ d	xistence and omain, co-d	l uniquene iomain and	s range									0													
			leniae 1.8 Ma	- 16							Instan 1.8	line If																									
	•																																				
	•																																				
	From	m Ma	atrices	to Fu	nction	s						Funct	ions fr	om C	alculus	5																					
	ı	Let A b	oe an m	$\times n$ mat	rix. We d	define a	function					Ма	ny of the	functio	ns we kn	iow have	domain	n and co	domain R	We can																	
				T	$: \mathbb{R}^n \rightarrow$	R <sup>m</sup> , 7	$\Gamma(\vec{v}) = A$	ï				exp	ress the	rule tha	f:R·	the fun $\rightarrow \mathbb{R}$	f(x) =	this way $sin(x)$																			
	T	≀hisis ∘Th	called a le domai	matrix t in of T is	ransforr s R <sup>n</sup> .	nation.						In o hor	calculus v izontal a	ve often xis is the	think of domain	a funct n, and ti	ion in te he vertic	rms of its al axis is	graph, wi	iose nain.																	
		• Th • Th	e co-do	main or $T(\vec{x})$ is	target o the imag	tTisR geof求i	under $T$									y . 1	$\sim$																				
	1	• In This giv	ves us air	all possi other in	iterpretat	tion of $A$	is the ran $4\vec{x} = \vec{b}$ :	ige.						_/		_/	$\sum$	_/	$\xrightarrow{\sin(x)}{x}$																		
		• set • au	of equa gmented	matrix											-7	1																					
		• ma • vec	ctor equa	ition formatio	n oquativ							Thi	is is ok w domain	then the is $\mathbb{R}^2$ ar	domain d the co	and cod	lomain a is R <sup>3</sup> . V	re IR. It's Ve would	hard to de need five	when																	
	Section 1.1			101110210	ii equilati						,	Girr	Side 58	to draw	that grap	pn.																					
	•																																				
	Ex	amp	le 1											Lin	ear T	ransf	orma	tions																			
		Let .	$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	1	$\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$	, $\vec{b} =$	7 5.								A func	tion $T$ $(\vec{n} \pm \vec{n})$	$: \mathbb{R}^n \rightarrow -T(i)$	$\mathbb{R}^m$ is $T \perp T$ (i	linear if	र्गन in l	<b>Q</b> n																
			[]	1]	L	1	[7]								• T	$(c\vec{v}) =$	$cT(\vec{v})$	for all $\bar{v}$	$f \in \mathbb{R}^n$ , a	nd $c$ in	R.																
		a) C	Comput	e $T(\vec{u})$ .											So if T	' is line τ	ar, the	n	a. #. )	$T(\vec{x})$	1	$- \alpha T$	(st.)														
							. ī								This is	called	the pri	nciple	of super	position	. Th	e idea i	s that	if we													
		b) C	alculat.	ev∈⊮	(* so th	iat T (v	b) = b								know 7	$\Gamma(\vec{e}_1), .$	$\dots, T(\bar{e})$	$f_n$ ), the	n we kno	w every	$T(\vec{v})$																
		c) G	Sive a $\bar{c}$	$f \in \mathbb{R}^3$ s	o there	is no i	ซี with 7	"( <i>v</i> ) =	$\vec{c}$						Fact:	Every n	natrix t	ransfor	mation T	A is line	ar.																
		0	r: Give	a č tha	it is not	t in the	e range	of T.	-1	-6.4																											
		0	: Give	a c tha	ic is not	, in the	: span o	i the C	oiumns	or A.																											
	Section	n 1.8 S	lide 60											Section :	.8 Side (	51												_		_							
-																																		-	-		-
	•								•																			•									
									•																								•			•	
																																	•			•	
			-	-	-	-						-		-		-	-									-	-	-	-								

# Example 2

Section 1.8 Slide 62

Suppose T is the linear transformation  $T(\vec{x})=A\vec{x}.$  Give a short geometric interpretation of what  $T(\vec{x})$  does to vectors in  $\mathbb{R}^2.$ 

1) 
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
  
2)  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$   
3)  $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$  for  $k \in \mathbb{R}$ 

Example 3

$$\begin{array}{l} \text{What does } T_A \text{ do to vectors in } \mathbb{R}^3?\\ \text{a)} \ A = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

b)  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 

Section 1.8 Slide 63

																•			
																. •			

E	xam	ple	3																								
	W	nat d	bes $T_{i}$	A do	to ve	ectors	in R	<sup>3</sup> ?																			
		) 4.	[1	0	0																			•			
	d	) A :		0	0														•		•	•					
			E1	0	o7																						
	b	) A =	= 0	-1	0																						
			[0	0	1																						
																						•					
																								•			
Secti	on 1.8	Slide 63																									
•	•						•	•	•		•	•	•	•	•				•		•	•					
•	•						•				•	•			•							•					
																								•			
																			•		•	•					
																										•	
•	•					•	•	•	•	•	•	•	•	•	•				•		•	•		•		•	
																								•			

# Example 4

A linear transformation  $T~:~\mathbb{R}^2\mapsto\mathbb{R}^3$  satisfies

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\2\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\8\\0\end{bmatrix}$$

What is the matrix that represents T?

# **1.8** EXERCISES

1. Let 
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
, and define  $T : \mathbb{R}^2 \to \mathbb{R}^2$  by  $T(\mathbf{x}) = A\mathbf{x}$ .  
Find the images under  $T$  of  $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ .  
2. Let  $A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$ ,  $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ , and  $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \\ c \end{bmatrix}$ .  
Define  $T : \mathbb{R}^3 \to \mathbb{R}^3$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Find  $T(\mathbf{u})$  and  $T(\mathbf{v})$ .

In Exercises 3–6, with T defined by  $T(\mathbf{x}) = A\mathbf{x}$ , find a vector  $\mathbf{x}$  whose image under T is  $\mathbf{b}$ , and determine whether  $\mathbf{x}$  is unique.

**3.** 
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$
  
**4.**  $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$   
**5.**  $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$   
**6.**  $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$ 

- 7. Let A be a  $6 \times 5$  matrix. What must a and b be in order to define  $T : \mathbb{R}^a \to \mathbb{R}^b$  by  $T(\mathbf{x}) = A\mathbf{x}$ ?
- 8. How many rows and columns must a matrix A have in order to define a mapping from  $\mathbb{R}^4$  into  $\mathbb{R}^5$  by the rule  $T(\mathbf{x}) = A\mathbf{x}$ ?

For Exercises 9 and 10, find all **x** in  $\mathbb{R}^4$  that are mapped into the zero vector by the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  for the given matrix A.

$$\mathbf{9.} \ A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

#### 70 CHAPTER 1 Linear Equations in Linear Algebra

18. The figure shows vectors u, v, and w, along with the images T(u) and T(v) under the action of a linear transformation T : R<sup>2</sup> → R<sup>2</sup>. Copy this figure carefully, and draw the image T(w) as accurately as possible. [*Hint:* First, write w as a linear combination of u and v.]



- **19.** Let  $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ , and  $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ , and let  $T : \mathbb{R}^2 \to \mathbb{R}^2$  be a linear transformation that maps  $\mathbf{e}_1$  into  $\mathbf{y}_1$  and maps  $\mathbf{e}_2$  into  $\mathbf{y}_2$ . Find the images of  $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$  and  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .
- **20.** Let  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,  $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ , and  $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ , and let  $T : \mathbb{R}^2 \to \mathbb{R}^2$  be a linear transformation that maps  $\mathbf{x}$  into  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ . Find a matrix A such that  $T(\mathbf{x})$  is  $A\mathbf{x}$  for each  $\mathbf{x}$ .

In Exercises 21 and 22, mark each statement True or False. Justify

$$\mathbf{10.} \ A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$$

 $\Gamma - 17$ 

**11.** Let  $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , and let A be the matrix in Exercise 9. Is  $\mathbf{b}$  in

the range of the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$ ? Why or why not?

12. Let 
$$\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$
, and let A be the matrix in Exercise 10. Is

**b** in the range of the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$ ? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot  $\mathbf{u} = \begin{bmatrix} 5\\2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2\\4 \end{bmatrix}$ , and their images under the given transfor-

mation T. (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what T does to each vector  $\mathbf{x}$  in  $\mathbb{R}^2$ .

**13.** 
$$T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
  
**14.**  $T(\mathbf{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$   
**15.**  $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$   
**16.**  $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ 

**17.** Let  $T : \mathbb{R}^2 \to \mathbb{R}^2$  be a linear transformation that maps  $\mathbf{u} = \begin{bmatrix} 5\\2 \end{bmatrix}$  into  $\begin{bmatrix} 2\\1 \end{bmatrix}$  and maps  $\mathbf{v} = \begin{bmatrix} 1\\3 \end{bmatrix}$  into  $\begin{bmatrix} -1\\3 \end{bmatrix}$ . Use the fact that *T* is linear to find the images under *T* of 3**u**, 2**v**, and 3**u** + 2**v**.

Make two sketches similar to Figure 6 that illustrate properties (i) and (ii) of a linear transformation.

- 24. Suppose vectors v<sub>1</sub>,..., v<sub>p</sub> span ℝ<sup>n</sup>, and let T : ℝ<sup>n</sup> → ℝ<sup>n</sup> be a linear transformation. Suppose T(v<sub>i</sub>) = 0 for i = 1,..., p. Show that T is the zero transformation. That is, show that if x is any vector in ℝ<sup>n</sup>, then T(x) = 0.
- **25.** Given  $\mathbf{v} \neq \mathbf{0}$  and  $\mathbf{p}$  in  $\mathbb{R}^n$ , the line through  $\mathbf{p}$  in the direction of  $\mathbf{v}$  has the parametric equation  $\mathbf{x} = \mathbf{p} + \mathbf{r} \mathbf{v}$ . Show that a linear transformation  $T : \mathbb{R}^n \to \mathbb{R}^n$  maps this line onto another line or onto a single point (a *degenerate line*).
- 26. Let u and v be linearly independent vectors in R<sup>3</sup>, and let P be the plane through u, v, and 0. The parametric equation of P is x = su + tv (with s, t in R). Show that a linear transformation T : R<sup>3</sup> → R<sup>3</sup> maps P onto a plane through 0, or onto a line through 0, or onto just the origin in R<sup>3</sup>. What must be true about T(u) and T(v) in order for the image of the plane P to be a plane?
- 27. a. Show that the line through vectors p and q in ℝ<sup>n</sup> may be written in the parametric form x = (1 − t)p + tq. (Refer to the figure with Exercises 21 and 22 in Section 1.5.)
  - b. The line segment from **p** to **q** is the set of points of the form  $(1-r)\mathbf{p} + r\mathbf{q}$  for  $0 \le t \le 1$  (as shown in the figure below). Show that a linear transformation T maps this line segment or to a line segment or onto a single point.



In Exercises 21 and 22, mark each statement True or False. Justify each answer.

- a. A linear transformation is a special type of function.
  b. If A is a 3 × 5 matrix and T is a transformation defined by T(x) = Ax, then the domain of T is R<sup>3</sup>.
  - c. If A is an  $m \times n$  matrix, then the range of the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is  $\mathbb{R}^m$ .
  - d. Every linear transformation is a matrix transformation.
  - e. A transformation T is linear if and only if  $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$  for all  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in the domain of T and for all scalars  $c_1$  and  $c_2$ .
- 22. a. Every matrix transformation is a linear transformation.
  - b. The codomain of the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is the set of all linear combinations of the columns of A.
    - c. If T : ℝ<sup>n</sup> → ℝ<sup>m</sup> is a linear transformation and if c is in ℝ<sup>m</sup>, then a uniqueness question is "Is c in the range of T?"
  - d. A linear transformation preserves the operations of vector addition and scalar multiplication.
  - e. The superposition principle is a physical description of a linear transformation.

**23.** Let  $T : \mathbb{R}^2 \to \mathbb{R}^2$  be the linear transformation that reflects each point through the  $x_1$ -axis. (See Practice Problem 2.)

**28.** Let **u** and **v** be vectors in  $\mathbb{R}^n$ . It can be shown that the set *P* of all points in the parallelogram determined by **u** and **v** has the form au + bv, for  $0 \le a \le 1$ ,  $0 \le b \le 1$ . Let  $T : \mathbb{R}^n \to \mathbb{R}^m$  be a linear transformation. Explain why the image of a point in *P* under the transformation *T* lies in the parallelogram determined by *T*(**u**) and *T*(**v**).

(t = 0) p

- **29.** Define  $f : \mathbb{R} \to \mathbb{R}$  by f(x) = mx + b.
  - a. Show that f is a linear transformation when b = 0.
  - b. Find a property of a linear transformation that is violated when  $b \neq 0$ .

c. Why is f called a linear function?

- **30.** An *affine transformation*  $T : \mathbb{R}^n \to \mathbb{R}^m$  has the form T(x) = Ax + b, with A an  $m \times n$  matrix and b in  $\mathbb{R}^m$ . Show that T is *not* a linear transformation when  $\mathbf{b} \neq \mathbf{0}$ . (Affine transformations are important in computer graphics.)
- **31.** Let  $T : \mathbb{R}^n \to \mathbb{R}^m$  be a linear transformation, and let  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  be a linearly dependent set in  $\mathbb{R}^n$ . Explain why the set  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$  is linearly dependent.

In Exercises 32–36, column vectors are written as rows, such as  $\mathbf{x} = (x_1, x_2)$ , and  $T(\mathbf{x})$  is written as  $T(x_1, x_2)$ .

**32.** Show that the transformation T defined by  $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$  is not linear.

	each	poin	it thro	ough	the 3	r <sub>1</sub> -ax	is. (S	ee P	ractic	e Pro	oblem	2.)	(4	$x_1 - $	$2x_2,$	$3 x_2 $	is no	t line	ar.													
							•													•												
																															•	
•	•			•	•							•							•			•		•			•	•			•	•
•	•	•	•	•	•		•					•						•	•	•	•	•		•	•		•	•	•		•	
•	•																	•			•										•	
							•											•		•	•				•						•	
																														•		
•	•						•											•		•	•										•	
							•													•										•	•	
																																0
																															•	0
		•	•																										•			
•	•						•					•							•	•		•									•	
•	•	•	•	•	•		•					•						•	•	•	•	•		•	•		•	•	•		•	
•	•																	•			•										•	
																									•					•		
				•	•		•											•		•	•			•	•		•	•			•	
																																0

	a a a a a a a a a a a <b>ILA</b> a	M Intowactiva Linoan Algobra
	1.9 : Matrix of a Linear Transformation	Dan Margalit, Joseph Rabinoff PDF version
Section 1.9 : Linear Transforms	Topics We will course these tanging in this costion	= index VIP Next >
Chapter 1 : Linear Equations	Ve will cover these topics in this section.  1. The standard vectors and the standard matrix.  2. Thus not three dimensional transformations in more detail	Interactive Linear Algebra
main 1334 Emeir Aigeora	Two and three dimensional transformations in more decan.     Onto and one-to-one transformations.	School of Mathematics Georgia Institute of Technology
$\begin{bmatrix} \cos 90^{\circ} \sin 90^{\circ} \end{bmatrix} \begin{bmatrix} \alpha . \end{bmatrix} = \begin{bmatrix} \infty & 0 \end{bmatrix}$	Objectives For the topics covered in this section, students are expected to be able to	Joseph Rabinoff School of Mathematics Georgia Institute of Technology
	do the following. 1. Identify and construct linear transformations of a matrix.	June 3, 2019
https://skcd.com/184	C. Unaracterize linear transformations as onto and/or one-to-one.     Solve linear systems represented as linear transforms.     Evences linear transforms in other forces such as as matrix equations	
	or as vector equations.	
tion 1.9 Steller 15	Section 1.3 Section 1.0	
CHECK OUT the textbook for Math 155	3 which was created by Georgia Tech professors for	
Intro. Linear Algebra		
https://textbooks.math.gatech.edu/ila/		
	noformations	///////////////////////////////////////
mere's a really nice section on linear tra	แรงกาลแขกร	K.////////////////////////////////////
		J / / / ¥ / / // 12 / Martin ( Dog )       · · ·
Transformations a presented. At this point it is convenient to fix our ideas and terminology regarding function		
At this point it is convenient to its our deal and commonly regarding function which we will call <i>transformations</i> in this book. This allows us to systematize ou discussion of matrices as functions.	https://textbooks.mat	th.gatech.edu/ila/one-
<b>Definition.</b> A transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each v	to-one-onto html	
<ul> <li>R<sup>*</sup> is called the <i>domain</i> of T.</li> </ul>		
<ul> <li>R<sup>n</sup> is called the <i>codomain</i> of T.</li> <li>For x in R<sup>n</sup>, the vector T(x) in R<sup>n</sup> is the <i>image</i> of x under T.</li> </ul>		
• The set of all images $\{T(x) \mid x \text{ in } \mathbb{R}^n\}$ is the <i>range</i> of <i>T</i> .	Example (Reflection). ^	
In a notation $T : \mathbf{K} \longrightarrow \mathbf{K}$ means $T$ is a transformation from $\mathbf{K}$ to $\mathbf{K}$ . It may help to think of $T$ as a "machine" that takes $x$ as an input, and gives you	T(x)	
as the output.	Let (-1.0)	
	$A = \left(\begin{array}{c} & & \\ 0 & & 1 \end{array}\right).$	
T	Describe the function $b = Ax$ geometrically.	
x.	In the equation $Ax = b$ , the input vector x and the output vector b are both in $x^2 - x^2$ .	
T(x) Tunge	$\mathbf{x}^*$ . First we multiply A by a vector to see what it does: $\begin{pmatrix} x \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix}$	
$\mathbf{R}^n \xrightarrow{T} \mathbf{R}^m$ domain codomain	$A\begin{pmatrix} y\\ y \end{pmatrix} = \begin{pmatrix} 0 & 1\\ 0 & 1 \end{pmatrix} \begin{pmatrix} y\\ y \end{pmatrix} = \begin{pmatrix} y\\ y \end{pmatrix}.$	https://
	Multiplication by A negates the x-coordinate: it reflects over the y-axis.	<u>Intp3.//</u>
Example (A matrix transformation that is neither one-to-one nor onto).	b = Ax	textbooks.math.gate
Let		ch edu/ila/matrix-
$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix},$		
and define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x) = Ax$ . This transformation is neither one-to- one nor onto, as we saw in this <u>example</u> and this <u>example</u> .		transformations.ntml
$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1.00 \\ 2.00 \end{bmatrix} = \begin{bmatrix} 3.00 \end{bmatrix}$	$\begin{bmatrix} 0.95 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} 2.00 \\ 4.00 \end{bmatrix} = \begin{bmatrix} 1.90 \\ 4.00 \end{bmatrix}$ xscale 0.95	
$\begin{bmatrix} -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 2.300 \\ 3.00 \end{bmatrix} = \begin{bmatrix} -6.00 \end{bmatrix}$ (Click and drag the beads of x and b)	[Click and drag the vector heads]	
input Output	xshear 0	
×	yshear 0 + Transform 2	
	Transform 3     Close Controls	
5.7 2 Y		
A picture of the matrix transformation T. The violet plane is the solution set of $T(x) = 0$ . If you drag x along the violet plane, the output $T(x) = Ax$ does not change This demonstrate that $T(x) = 0$ has an entry of the solution.	5.7	
not sharpe. Ints aemonstrates that $T(x) \equiv 0$ has more than one solution, so T is not one-to-one. The range of T is the violet line on the right; this is smaller than the codomain $\mathbb{R}^2$ . If you drag b off of the violet line, then the	Multiplication by the matrix A reflects over the y-axis. Move the input vector	
equation $Ax = b$ becomes inconsistent; this means $T(x) = b$ has no solution.	x to see how the output vector b changes.	
		** * * * * * * * * * *

															-					We	ek Dat	es	Lectu	re	Studi	0		Lecture	•		Stu	dio	L	ecture
		Se	ection	1.9 :	Linear	Transf	orms				1.9	: Mat	trix of	a Line	ear Ira	ansforr	nation			1	1/8	- 1/12	1.1		WS1	1		1.2			WS	1.2	1	.3
			Ch	apter 1 :	Linear Eq	uations						We will 1. The	cover the	se topic rd vecto	s in this <b>ors</b> and t	section. he <b>stand</b>	ard mat	rix.		2	1/1	5 - 1/19	Break		WS1	.3		1.4			WS	1.4	1.	.5
			М	lath 1554	Linear Alı	gebra						2. Tw 3. On	o and thr to and o	ee dime ne-to-o	nsional t ne transf	ransform formation	ations in 15.	more det:	ail.	3	1/2	2 - 1/26	1.7		WS1	.5,1.7		1.8			WS	1.8	1	.9
			[ cos	90° 50 90	][a.]							Objecti For the	ves topics co	vered in	this sect	ion, stud	ents are	expected	to be able	1 4	1/2	9 - 2/2	1.9,2.	1	WS1	9,2.1		Exam 1	, Revie	w	Car	ncelled	2	.2
			l-sin	90° (rs 90°	[[a;] =	22						do the f 1. Ide	ollowing. ntify and	constru	ct linear	transform	nations o	f a matrix																
				https://x	kcd.com/	184						2. Chi 3. Sol	aracterize ve linear	linear t systems v. transf	ransform represen	ations as ted as lir	onto and lear trans	i/or one-t forms.	o-one.	or									1.11				r i	
												or a	as vector	equation	is.	Juner Torr	ns, such .	is as mai	rix equatio	ns									HA		-	-		
stice 1.)	9 S54+85										Section	1.9 Side 66														/			-		-			
																											19	1		19		-		
																										a	1	2	The second					
																									ſ			RI	DAY	7	A.	S		
	Defi	nitior	n: Th	ne Sta	ndarc	l Vect	tors							А	Prop	ertv c	of the	Stand	lard V	ectors								1.500			makear	nemelorg		
	1	he sta	ndard	vectors	in R <sup>n</sup>	are the	vectors	5 ē <sub>1</sub> , ē <sub>2</sub> ,	, $\vec{e}_n$ ,	, where					Note	: if A is	an ma	< n mat	rix with (	olumns	$\vec{v}_1, \vec{v}_2, .$	$\ldots, \vec{v}_n$ , th	en											
				$\vec{e}_1 =$		$\vec{e}_2$	-		$\vec{e}_n =$									$A\vec{e}_i =$	$\vec{v}_i$ , for $i$	= 1, 2,	,n												•	•
															So mu	ultiplyin	g a mat	trix by $\vec{e}$	gives co	olumn i	of $A$ .													
															Exam	(1	2 3																	•
	F	or exar	nple, ir	n ℝ <sup>3</sup> ,												$\begin{pmatrix} 4 \\ 7 \end{pmatrix}$	$\binom{5}{8} \binom{6}{9}$	$\vec{e}_{2} =$																
				$e_1 =$		$e_2$			$e_3 =$																									
																																	•	•
	Section 1.5	Side 67												Secto	1.9 Sid	e 68																	•	•
																																	•	
																																•	•	
																									•									
																					•	• •			• •									•
	•	•													•						•	• •			• •							•		
•																									•									
																					•	• •			• •								•	•
																					•	• •			• •								•	•
•																						• •			• •									
																					•				• •							•	•	
																					•				• •							•	•	
																					•				• •							•	•	
																						• •			• •								•	•
																						• •			• •								•	•
																									• •									
																									• •									
																									• •									

# The Standard Matrix

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a is a unique matrix $A$ such	) linear transformation. Then there ch that
$T(\vec{x}) =$	$A\vec{x},  \vec{x} \in \mathbb{R}^m.$
In fact, $A$ is a $m \times n,$ and	d its $j^{th}$ column is the vector $T(\vec{e}_j)$ .
$A = \left[T(\vec{e}_1)\right]$	$T(\vec{e}_3) \cdots T(\vec{e}_n)$

The matrix A is the standard matrix for a linear transformation.

## Rotations

Example 1 What is the linear transform  $T:\mathbb{R}^2\to\mathbb{R}^2$  defined by

at is the initial dataset in a fact of a defined by

 $T(\vec{x})=\vec{x}$  rotated counterclockwise by angle  $\theta?$ 

	Section 3	1.9 S	ide 69									Section 1.9	Slide	10									
•																		•					
																		•					
																		•					
																		•					
																		•					
																		•					
																		•					
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
																		•					
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
																		•					
											•		•	•				•			•		
											•		•	•				•			•		
																					•		
																					•		
																					•		

# The Standard Matrix

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a is a unique matrix $A$ such	) linear transformation. Then there ch that
$T(\vec{x}) =$	$A\vec{x},  \vec{x} \in \mathbb{R}^m.$
In fact, $A$ is a $m \times n,$ and	d its $j^{th}$ column is the vector $T(\vec{e}_j)$ .
$A = \left[T(\vec{e}_1)\right]$	$T(\vec{e}_3) \cdots T(\vec{e}_n)$

The matrix A is the standard matrix for a linear transformation.

## Rotations

Example 1 What is the linear transform  $T:\mathbb{R}^2\to\mathbb{R}^2$  defined by

at is the initial dataset in a fact of a defined by

 $T(\vec{x})=\vec{x}$  rotated counterclockwise by angle  $\theta?$ 

	Section 3	1.9 S	ide 69									Section 1.9	Slide	10									
•																		•					
																		•			•		
																		•					
																		•					
																		•					
																		•					
																		•					
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
																		•					
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
									•	•					•	•							
																		•					
											•		•	•				•			•		
											•		•	•				•			•		
																					•		
																					•		
																					•		

								•	•								0					•	•	0		•	• •					•			•	•		
																											• •											
																											• •											
																							•	•		•										•		
																											• •											
																								•			• •											
																							•			•										•		
																							•														-	
																											• •											
1       1																																						
.       .								•	•					0			0						•	0		•	• •					•				•		
1       1																																						
.       .																																						
1       1																											• •											
1       1																																						
1       1																																						
1       1																																						
.       .																																						
	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		-			-	-	-	-	-	-	-	-	-	-	-
· · · · · · · · · · · · · · · · · · ·										•														•			• •							•			•	
· · · · · · · · · · · · · · · · · · ·										•													•	•		•	• •						•	•		•		
								•	•														•	•		•	• •					•		•		•		
								•	•			•									•											•		•		•		

Ē	Ēx	. L	_e	t Ț	- (x	)=	A,	< k	De	ť	ne	tra	an	sf	or	m	at	ioi	า	•	•	1	The	e Sta	anda	rd N	latrix	¢									
۰V	vh	ic	h	fir	st	re	efl∈	ec	ts	Ve	ЭC	to	rs	in	F	<u>^</u> 2	2.	لديم		•	•			ſ	Let is a	orem T : R unique	$n \mapsto 1$ matr	R <sup>m</sup> be ix A s	a line uch th	ear tra nat	insforr	natior	n. The	n ther	e		
Ę		0	SS	τr	ne	IIr	1e	y:	=0	), a	an	a	τņ	en	ı p	orc	)je	CI	S									$T(\vec{x})$	$= A\vec{x}$	ith i	$\vec{x} \in \mathbb{R}$	m.		(T)			
_t	he	) r	es	sul	tir	Ŋ	Ve	ЭÇ	to	r t	0	th	e	y-a	ax	iş.	• .								In fa	ct, A	sam A=	$\times n$ , a	) T	j <sup>™</sup> co (∉_)	Jumn	is the $T(\vec{e}_{r})$	vector	$T(e_j)$			
																								l				[1 (0)	, 1	(c3)		r (0n)	1				
F	in	d	ťh	ne	st	ar	nda	ar	ď	ma	atr	rix	Ö	f A	١.	•	•	•	•	•	•			The n	natrix	A is t	he <b>sta</b>	ndard	mat	rix for	a line	ar tra	nsform	nation	-		
																							Section 1	9 Sid	le 69												
																																		•			•
						•										•									•					•			•				
																																				•	
																														•				•		•	
																					•						•		•			•	•				•
	•					•										•					•				•		•		•			•	•				
																																					•
						•										•									•					•				•		•	•
	•					•		•					•			•	•	•						•	•				•			•	•	•	•		
-	-	,	-	-	-		,	-	-	-	-	-	-		-		-	-		-	-		,	-			-	-	-	-	-	-	-	-	-	-	-

### Onto

Definition A linear transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  is **onto** if for all  $\vec{b} \in \mathbb{R}^m$  there is a  $\vec{x} \in \mathbb{R}^n$  so that  $T(\vec{x}) = \vec{b}$ .

Onto is an existence property: for any  $\vec{b} \in \mathbb{R}^m$ ,  $A\vec{x} = \vec{b}$  has a solution.

#### Examples

- · A rotation on the plane is an onto linear transformation.
- A projection in the plane is not onto.

#### Useful Fact

T is onto if and only if its standard matrix has a pivot in every row.

a port in every Row A has <=> PREF of A has no zero rows.

Section 1.9 Slide 79

## One-to-One

Definition

```
A linear transformation T : \mathbb{R}^n \to \mathbb{R}^m is one-to-one if
for all \vec{b} \in \mathbb{R}^m there is at most one (possibly no) \vec{x} \in \mathbb{R}^n so
that T(\vec{x}) = \vec{b}.
```

One-to-one is a uniqueness property, it does not assert existence for all  $\vec{b}$ .

#### Examples

- · A rotation on the plane is a one-to-one linear transformation.
- · A projection in the plane is not one-to-one.

### Useful Facts

on 1.9

- T is one-to-one if and only if the only solution to  $T\left(\vec{x}\right)=0$  is the zero vector,  $\vec{x} = \vec{0}$ .
- T is one-to-one if and only if the standard matrix A of T has no free variables.

Slure ac  $\Rightarrow$ has a pivotik every could A Q: Example of transformation which is (> Ax=6 has at most one soluti (G) one-to-one but not onto? one-to-a (b) 01+1

#### Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of  $\mathbb{R}^2$  in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, ...)
- · Please familiarize yourself with them: you are expected to memorize them (or be able to derive them)

#### The Standard Matrix

$\fbox{\begin{tabular}{c} \hline Theorem \\ Let $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix $A$ such that \end{tabular}$
$T(\vec{x}) = A\vec{x},  \vec{x} \in \mathbb{R}^{m}.$
In fact, $A$ is a $m \times n,$ and its $j^{th}$ column is the vector $T(\vec{e}_j).$
$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_3}) & \cdots & T(\vec{e_n}) \end{bmatrix}$



Two Dimensional Examples: Reflections

Section 1.9 Slide 72

The matrix A is the standard matrix for a linear transformation.

#### Two Dimensional Examples: Reflections



### Two Dimensional Examples: Contractions and Expansions

transformation	image of unit square	standard matrix
Vertical Contraction	$\vec{e_2}$ $\vec{e_2}$ $\vec{e_1}$ $\vec{x_1}$	$\begin{pmatrix} 1 & 0\\ 0 & k \end{pmatrix},  k  < 1$
Vertical Expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \ k > 1$

### Two Dimensional Examples: Shears





### Two Dimensional Examples: Shears



# Two Dimensional Examples: Projections



Section 1.9 Slide 77

Section 1.9 Slide 78

### Example

Complete the matrices below by entering numbers into the missing entries so that the properties are satisfied. If it isn't possible to do so, state why.

a) A is a  $2\times 3$  standard matrix for a one-to-one linear transform  $A=\left( \begin{array}{cc} 1&0\\ 0&1 \end{array} \right)$ 

b) B is a  $3 \times 2$  standard matrix for an onto linear transform.

 $B = \begin{pmatrix} 1 \\ & \end{pmatrix}$ 

c) C is a  $3\times 3$  standard matrix of a linear transform that is one-to-one and onto.



Theorem

For a linear transformation  $T~:~\mathbb{R}^n\to\mathbb{R}^m$  with standard matrix A these are equivalent statements.

- 1. T is onto.
- 2. The matrix A has columns which span  $\mathbb{R}^m$
- 3. The matrix A has m pivotal columns.

#### Theorem -

For a linear transformation  $T:\mathbb{R}^n\to\mathbb{R}^m$  with standard matrix A these are equivalent statements.

- 1. T is one-to-one.
- 2. The unique solution to  $T(\vec{x}) = \vec{0}$  is the trivial one.
- 3. The matrix A linearly independent columns.
- 4. Each column of A is pivotal.

Section 1.9	Slide 82

# Example 2 Define a linear transformation by

 $T(x_1,x_2)=(3x_1+x_2,5x_1+7x_2,x_1+3x_2).$  Is this one-to-one? Is it onto?

### Additional Example (if time permits)

Let  ${\boldsymbol{T}}$  be the linear transformation whose standard matrix is

	1	0	0
4	-4	8	1
A =	2	$^{-1}$	3
	0	0	5

Is the transformation onto? Is it one-to-one?

Section 1.9 Slide 83

Section 1.9 Slide 84

# 1.9 EXERCISES

In Exercises 1–10, assume that T is a linear transformation. Find the standard matrix of T.

- **1.**  $T : \mathbb{R}^2 \to \mathbb{R}^4$ ,  $T(\mathbf{e}_1) = (3, 1, 3, 1)$  and  $T(\mathbf{e}_2) = (-5, 2, 0, 0)$ , where  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$ .
- **2.**  $T : \mathbb{R}^3 \to \mathbb{R}^2$ ,  $T(\mathbf{e}_1) = (1, 3)$ ,  $T(\mathbf{e}_2) = (4, -7)$ , and  $T(\mathbf{e}_3) = (-5, 4)$ , where  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are the columns of the  $3 \times 3$  identity matrix.
- **3.**  $T : \mathbb{R}^2 \to \mathbb{R}^2$  rotates points (about the origin) through  $3\pi/2$  radians (counterclockwise).
- 4.  $T : \mathbb{R}^2 \to \mathbb{R}^2$  rotates points (about the origin) through  $-\pi/4$  radians (clockwise). [*Hint:*  $T(\mathbf{e}_1) = (1/\sqrt{2}, -1/\sqrt{2}).]$
- T: ℝ<sup>2</sup> → ℝ<sup>2</sup> is a vertical shear transformation that maps e<sub>1</sub> into e<sub>1</sub> − 2e<sub>2</sub> but leaves the vector e<sub>2</sub> unchanged.
- T: ℝ<sup>2</sup> → ℝ<sup>2</sup> is a horizontal shear transformation that leaves
   e<sub>1</sub> unchanged and maps e<sub>2</sub> into e<sub>2</sub> + 3e<sub>1</sub>.
- 7.  $T : \mathbb{R}^2 \to \mathbb{R}^2$  first rotates points through  $-3\pi/4$  radian (clockwise) and then reflects points through the horizontal  $x_1$ -axis. [*Hint*:  $T(\mathbf{e}_1) = (-1/\sqrt{2}, 1/\sqrt{2})$ .]
- 8.  $T : \mathbb{R}^2 \to \mathbb{R}^2$  first reflects points through the horizontal  $x_1$ -axis and then reflects points through the line  $x_2 = x_1$ .
- 9. T: ℝ<sup>2</sup> → ℝ<sup>2</sup> first performs a horizontal shear that transforms e<sub>2</sub> into e<sub>2</sub> 2e<sub>1</sub> (leaving e<sub>1</sub> unchanged) and then reflects points through the line x<sub>2</sub> = -x<sub>1</sub>.
- **10.**  $T : \mathbb{R}^2 \to \mathbb{R}^2$  first reflects points through the vertical  $x_2$ -axis and then rotates points  $\pi/2$  radians.
- 11. A linear transformation T : ℝ<sup>2</sup> → ℝ<sup>2</sup> first reflects points through the x<sub>1</sub>-axis and then reflects points through the x<sub>2</sub>-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?
- **12.** Show that the transformation in Exercise 8 is merely a rotation about the origin. What is the angle of the rotation?
- Let T : ℝ<sup>2</sup> → ℝ<sup>2</sup> be the linear transformation such that T(e<sub>1</sub>) and T(e<sub>2</sub>) are the vectors shown in the figure. Using the figure, sketch the vector T(2, 1).



14. Let T: R<sup>2</sup> → R<sup>2</sup> be a linear transformation with standard matrix A = [a<sub>1</sub> a<sub>2</sub>], where a<sub>1</sub> and a<sub>2</sub> are shown in the figure. Using the figure, draw the image of [-1] under the



In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.



In Exercises 17–20, show that T is a linear transformation by finding a matrix that implements the mapping. Note that  $x_1, x_2, \ldots$  are not vectors but are entries in vectors.

- **17.**  $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$
- **18.**  $T(x_1, x_2) = (2x_2 3x_1, x_1 4x_2, 0, x_2)$
- **19.**  $T(x_1, x_2, x_3) = (x_1 5x_2 + 4x_3, x_2 6x_3)$
- **20.**  $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 4x_4$   $(T : \mathbb{R}^4 \to \mathbb{R})$
- **21.** Let  $T : \mathbb{R}^2 \to \mathbb{R}^2$  be a linear transformation such that  $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$ . Find **x** such that  $T(\mathbf{x}) = (3, 8)$ .
- **22.** Let  $T : \mathbb{R}^2 \to \mathbb{R}^3$  be a linear transformation such that  $T(x_1, x_2) = (x_1 2x_2, -x_1 + 3x_2, 3x_1 2x_2)$ . Find **x** such that  $T(\mathbf{x}) = (-1, 4, 9)$ .

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

- 23. a. A linear transformation T : ℝ<sup>n</sup> → ℝ<sup>m</sup> is completely determined by its effect on the columns of the n × n identity matrix.
  - b. If  $T : \mathbb{R}^2 \to \mathbb{R}^2$  rotates vectors about the origin through an angle  $\varphi$ , then T is a linear transformation.
  - c. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.
  - d. A mapping  $T : \mathbb{R}^n \to \mathbb{R}^m$  is onto  $\mathbb{R}^m$  if every vector **x** in  $\mathbb{R}^n$  maps onto some vector in  $\mathbb{R}^m$ .
  - e. If A is a  $3 \times 2$  matrix, then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  cannot be one-to-one.
  - a. Not every linear transformation from ℝ<sup>n</sup> to ℝ<sup>m</sup> is a matrix transformation.
    - b. The columns of the standard matrix for a linear transformation from R<sup>n</sup> to R<sup>m</sup> are the images of the columns of the n × n identity matrix.

#### 80 CHAPTER 1 Linear Equations in Linear Algebra

- c. The standard matrix of a linear transformation from  $\mathbb{R}^2$ to  $\mathbb{R}^2$  that reflects points through the horizontal axis, the vertical axis, or the origin has the form  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ , where a and d are  $\pm 1$ .
- d. A mapping  $T : \mathbb{R}^n \to \mathbb{R}^m$  is one-to-one if each vector in  $\mathbb{R}^n$  maps onto a unique vector in  $\mathbb{R}^m$ .
- e. If A is a 3 × 2 matrix, then the transformation x → Ax cannot map R<sup>2</sup> onto R<sup>3</sup>.

In Exercises 25–28, determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify each answer.

- 25. The transformation in Exercise 17
- **26.** The transformation in Exercise 2
- 27. The transformation in Exercise 19
- 28. The transformation in Exercise 14

In Exercises 29 and 30, describe the possible echelon forms of the standard matrix for a linear transformation T. Use the notation of Example 1 in Section 1.2.

- **29.**  $T : \mathbb{R}^3 \to \mathbb{R}^4$  is one-to-one.
- **30.**  $T : \mathbb{R}^4 \to \mathbb{R}^3$  is onto.
- 31. Let T: ℝ<sup>n</sup> → ℝ<sup>m</sup> be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T is one-to-one if and only if A has \_\_\_\_\_ pivot columns." Explain why the statement is true. [*Hint*: Look in the exercises for Section 1.7.]

32. Let T: R<sup>n</sup> → R<sup>m</sup> be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T maps R<sup>n</sup> onto R<sup>m</sup> if and only if A has \_\_\_\_\_\_ pivot columns." Find some theorems that explain why the statement is true.

**33.** Verify the uniqueness of *A* in Theorem 10. Let  $T : \mathbb{R}^n \to \mathbb{R}^m$  be a linear transformation such that  $T(\mathbf{x}) = B\mathbf{x}$  for some

 $m \times n$  matrix B. Show that if A is the standard matrix for T, then A = B. [Hint: Show that A and B have the same columns.]

- **34.** Why is the question "Is the linear transformation T onto?" an existence question?
- **35.** If a linear transformation  $T : \mathbb{R}^n \to \mathbb{R}^m$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^m$ , can you give a relation between *m* and *n*? If *T* is one-to-one, what can you say about *m* and *n*?
- 36. Let S : ℝ<sup>p</sup> → ℝ<sup>n</sup> and T : ℝ<sup>n</sup> → ℝ<sup>n</sup> be linear transformations. Show that the mapping x ↦ T(S(x)) is a linear transformation (from ℝ<sup>p</sup> b or ℝ<sup>n</sup>).[*Hint*: Compute T/S(cu + d x)) for u, v in ℝ<sup>p</sup> and scalars c and d. Justify each step of the computation, and explain why this computation gives the desired conclusion.]

[M] In Exercises 37–40, let T be the linear transformation whose standard matrix is given. In Exercises 37 and 38, decide if T is a one-to-one mapping. In Exercises 39 and 40, decide if T maps  $\mathbb{R}^5$ onto  $\mathbb{R}^5$ . Justify your answers.

0

	-5	10	-5	4			7	5	4	
37	8	3	-4	7		38	10	6	16	•
57.	4	-9	5	-3		50.	12	8	12	
	3	$^{-2}$	5	4			8	-6	-2	
		-		-	- 7					
	4	-7	3	1	5					
	6	-8	5	12	-8					
39.	-7	10	-8	-9	14					
	3	-5	4	2	-6					
	5	6	-6	-7	3					
	F 0	10	-	1						
	9	13	5	6	-1					
	14	15	-7	-6	4					
40.	-8	-9	12	-5	-9					
	-5	-6	-8	9	8					
	13	14	15	2	11					
	-				_					

1															 5		. 1														
																											•				
																														•	
																											•	•		•	
•																							•	•	•		•				
				•																	•						•			•	
				•																	•				•			•	•		
																											•				
																													•		
																						•			•						
																									•						
																	•										•			•	
																									•		•				