Handwritten Homework Assignments - Exploration for MATH 1554

For each assignment, complete the questions on a separate sheet of paper and put your name on it. Write neatly and use complete sentences where necessary. You must submit original work, but I'm okay with you all working together to share ideas. Handwritten homework assignments are due on Fridays. Please turn in handwritten homework in Gradescope (access the first time through Canvas) every Friday by midnight.

For 8/21 [NOT GRADED - PRACTICE ONLY!] In $\mathbb{R}^3$ (so using three coordinate axes) sketch the following, making a new sketch for each part (i)-(v):

(i) the plane $z = 0$,

(ii) the plane $z = 2$,

(iii) the plane $y = -3$,

(iv) the plane $x + y + z = 0$,

(v) the intersection of the planes $z = 2$ and $x = 0$.

In each case, you are practicing drawing an accurate, representative graph of the plane of points which satisfy the given equation in the variables x, y, and z.

For 8/28: (1) Choose two vectors v_1, v_2 in $\mathbb{R}^2$ and another vector b also in $\mathbb{R}^2$. Find scalars x, y in $\mathbb{R}$ such that $xv_1 + yv_2 = b$ (if this is not possible, pick other v_1, v_2, b vectors). Illustrate the vector equation you just solved by graphing the vectors v_1, v_2, b in the $x-y$-plane, and be sure to illustrate how b is obtained by adding a scalar of one vector to the other. (2) Repeat part 1 with vectors v_1, v_2, b in $\mathbb{R}^3$ that give a consistent system. (3) Why is part 2 more difficult than part 1? Explain clearly using complete sentences.

NOTE: For full credit, your answer must satisfy the following constraints: v_1, v_2 are not allowed to be scalar multiples of $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$; neither of the values of x, y is allowed to be $-1, 0$, or 1 for $\mathbb{R}^2$, and you can have at most one 0, and one 1 of these for R^3. The vectors v_1, v_2 should be "relatively random looking", subject to my discretion. Don't overthink part (3): a single sentence will suffice.

For 9/4: See Announcement in Canvas for MATLAB #1 Exploration.

NOT for 9/4 *Optional not graded - do not turn in - for practice exploration only - this week we are doing MATLAB (see entry above) - if you are interested you can do the following but it is not graded and you shouldn't turn it in Gradescope or anywhere else - the following is not graded - don't do the following only do MATLAB this week:* For each part, if possible, give an example *(if you want to but don't turn it in anywhere - this will not be graded - you need to do the MATLAB exploration this week - keep reading if you want but this is not graded and don't turn it in anywhere - do not email this to me please - for practice only)* of two sets A and B of vectors $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^m$ where the set A is **linearly independent** and the set B is **linearly dependent**, and if it is not possible to do so for either A or B explain why in your own words.

1. One vector in $\mathbb{R}^2$,

2. two vectors in $\mathbb{R}^2$,

3. three vectors in $\mathbb{R}^2$,

4. two vectors in $\mathbb{R}^3$,

5. three vectors in $\mathbb{R}^3$,

6. four vectors in $\mathbb{R}^3$.

(The above is not graded and shouldn't be submitted to Gradescope - just for practice only - if you want to come up with minimal examples of the above that's ok since I'm not grading it anyway - ONLY MATLAB IS GRADED/SUBMITTED THIS WEEK!)

For 9/11: For each matrix A below, (0) state the domain and codomain of T_A, (1) find $T_A(e_1), T_A(e_2)$, (2) find $T_A(v), T_A(w)$, (3) describe in a few words what the transformation is doing, and (4) determine if the associated transformation is one-to-one/onto or not, and (5) give the matrix an appropriate "name" (fine to be silly name like, e.g., "the x-zero-er" for projection to y-axis"). For the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

1. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

9. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, for the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

For 9/18: For full credit, correctly indicate which problem you are solving by writing the statement you are answering (like "$AB = 0$ and $A \neq 0, B \neq 0$"). For grading purposes, please try to write the problems in the same order as listed here. The matrix 0 is the zero matrix and the matrix I is the identity matrix.

For each problem find square matrices which satisfy the given conditions. You don't have to justify *how* you found the matrices for each problem, but you *must verify the equality with calculations* in each case. Just show the matrices A, B, C and the given products.

The following restrictions are required for each problem:
No matrix A, B, or C can be diagonal, none can be equal or a scalar multiple of each other, and no product can be the zero matrix (except *(d)*) or scalar multiple of the identity matrix (except *(e)*). All of the below are possible with these restrictions.

(a) $AB = BA$ but neither A nor B is 0 nor I, and $A \neq B$.

(b) $AB \neq BA$.

(c) $AB = AC$ but $B \neq C$, and the matrix A has *no zeros entries.*

(d) $AB = 0$ but neither A nor B is 0.

(e) $AB = I$ but neither A nor B is I.

For 9/25: MATLAB #2. See Announcement in Canvas.

NOT for 9/25: *Optional not graded - do not turn in - for practice exploration only - this week we are doing MATLAB (see entry above) - if you are interested you can do the following but it is not graded and you shouldn't turn it in Gradescope or anywhere else - the following is not graded - don't do the following only do MATLAB this week:*

Step 1: Pick a matrix and find nul(A). Pick a matrix A of size no smaller than 3×5 (to get a good feel for the problem). Choose entries not all positive, and not too many zeros, and your matrix shouldn't be rref (ideally, but it's ok to pick a matrix in rref if you want). Find the null space nul(A) by finding the parametric vector form of the general solution x to $A\mathrm{x} = 0$, and use these vectors to express nul(A) as their span. Call the vectors $v_1, \ldots, v_n$.

Step 2: An example that nul(A) is closed under vector addition. Choose two vectors $\mathrm{w}_1, \mathrm{w}_2$ in the span nul(A) $= \mathrm{span}\{v_1, \ldots, v_n\}$ from step 1. Do this by taking two or more vectors in the basis from step 1 and adding them to each other using some scalars, *i.e.* chose a random linear combination of the vectors from step 1. Do this twice, once to get w_1 and once to get w_2. Add these vectors together to get $z = \mathrm{w}_1 + \mathrm{w}_2$. Check that z is in the null space of A by verifying $Az = 0$.

Step 3: An example that nul(A) is closed under scalar multiplication. Chose a vector w in the null space of A. Choose a random scalar c. Check that cw is in the null space of A by verifying $Aw = 0$.

Step 4: The general case. Try to convince yourself that no matter how A is chosen, nul(A) is always closed under scalar multiplication and vector addition. The hint is that $A(x + y) = Ax + Ay$ and $A(cx) = c(Ax)$.

(The above is not graded and shouldn't be submitted to Gradescope - just for practice only - if you want to come up with minimal examples of the above that's ok since I'm not grading it anyway - ONLY MATLAB IS GRADED/SUBMITTED THIS WEEK!)

For 10/2: For the following exploration, your matrix A needs to be relatively random looking. So, in particular it should have:

- not too many 1's or 0's,
- not be in REF or RREF,
- not be a scalar multiple of the identity matrix, or have too many columns which are scalar multiples of e_i the standard basis vectors, and additionally
- should not have a very ugly answer to *(Part i)*.

Part i) Write down a system of three equations in three variables, whose augmented matrix $[A|b]$ would then be 3×4. Find the determinant of the matrix A whose entries are the coefficients of the system. Ensure the determinant is non-zero and find the inverse of the matrix and also calculate $A^{-1}b$ *(you may use matlab or an online calculator for this part).* Describe how $A^{-1}b$ relates to the system $[A|b]$ in a few words.

Part ii) For the matrix A you wrote down in *part (i)*, row reduce the matrix to rref. What is the rref of A? Following the row operations you made to reduce A to rref, state the determinant of each elementary matrix. That is, if I is the rref of A and $A \sim E_1A \sim E_2E_1A \sim \cdots \sim E_n \cdots E_2E_1A = I$ are the matrices you got when you row reduced A to I, then calculate or otherwise find the determinant det A, det E_1, det E_2, …, det E_n, and det I.

Part iii) Compute the determinant of each of the intermediate matrices E_1A, E_2E_1A, …, $E_n \cdots E_2E_1A$. Compare your result to what you did in *part (ii)*. Try to state the relationship between A and the intermediate row equivalent matrices, in terms of the determinants of the E_i's. Formulate a relationship between det(A) and the product of the det(E_i)'s.

For 10/9 Two separate parts.

1. Find a 2×2 matrix A which has **no real eigenvalues** and show that your answer is correct by finding the characteristic polynomial and explaining why it has no real roots.

2. Find all eigenvalues and a corresponding eigenvector for each matrix below **without calculations** by thinking it out using the linear transformation's **geometric interpretation**. Write a few words (like 5) in each case explaining why your eigenvector/eigenvalue pair works. For each matrix, graph each eigenvector and it's image after the transformation as well as a random NON-eigenvector. Check your graph is accurate by showing the matrix multiplication.

   (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A = id$.

   (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, T_A =projection onto x-axis.

   (c) $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, T_A =rotation by θ.
   (and state the values of θ for which A has eigenvectors)

   (d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, T_A =reflect about the line "$y = x$".

   (e) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, T_A =stretch in x-direction.

   (f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, T_A =shear.

   (g) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, T_A =stretch z-axis.

   (h) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, T_A =rotate by 90° about the x-axis.

For 10/16: **Part 0:** Create one **easy** and one **medium difficulty** exam or quiz style problem using the concept of diagonalization, or the concept of complex eigenvalues. You can pick only one topic, or one problem for each topic; it's totally up to you. You can write any combination of a true/false problem, a possible/impossible, an example construction, a computational problem, or two of the same kind of problem, but your two problems must be using only concepts taught in this course and must not be a problem on one of the practice/sample quizzes/exams. You must also **solve** your problem and state the solution, with a few words of justification, but you do not need to provide an elaborate or detailed solution. Your grade for this portion of the exploration will be based on how well you follow the directions in this paragraph. Don't over think it. Two simple problems will do.

Part 1: Show that there is no relationship between any kind of row operation and the eigenvalues of the matrices involved as follows. For each of the **three types** of row operation $cR_i + R_j \to R_j$, $cR_j \to R_j$, and $R_i \leftrightarrow R_j$ which are *adding two rows*, *multiplying a row by a scalar*, and *switching two rows*: Find four matrices (for a **total of 12 matrices**) A, B, C, D such that $A \neq B$ and $C \neq D$ and also $A \sim B$ and $C \sim D$, the matrix A is row equivalent to via exactly **one** row operation of the appropriate type B and also C is row equivalent to D via exactly **one** row operation of the appropriate type, and such that A and B have the exact same eigenvalues but C and D have different eigenvalues, and **be sure to state all eigenvalues**. Conclude that there is no relationship whatsoever between two matrices "being row equivalent" and "have the same eigenvalues".

Part 2: Let λ be a real eigenvalue of a matrix with real entries A. Show that the set $V_\lambda = \{x \ : \ Ax = \lambda x\}$ is a subspace of $\mathbb{R}^n$. If you reduce your solution to a question about null spaces, be sure to include prove that null spaces are subspaces (but that's fine if you want to do it that way so long as your argument is clear, and correct of course).
Hint: check the definition of subspace.

Part 3: Why is the set of eigenvectors of A corresponding to eigenvalue λ NOT a subspace? Why does this not contradict what you were asked to do in Part 2?
Hint: check the definition of eigenvector.

For 10/23: **Part 0:** *Optional:* Create two **easy** or **medium difficulty** exam or quiz style problems using any of the concepts from the previous two weeks (check the schedule to determine the topics). Include a solution and follow the same instructions from last week.

Part 1: NOTE: This week's assignment doesn't have to be handwritten. (For 1pt) Write a paragraph which explains how eigenvectors/eigenvalues or some other topic from the course are used in a field which interests you. Be specific and put some thought into this. (For 2pt) If your paragraph is not essentially the thing on wikipedia about how bridges have something to do with eigenvalues, but actually give some details or original content then you get 2pts instead of 1pt. (For 3pt) If you also support your research with real math or alternately something creative. This has to include *some mathematical content* but can take *any form whatsoever*. Last year's submissions included a poem, several posters, a few slide-show presentations like using PowerPoint, etc., and quitte a few of them actually were pretty decent research project results that I was quite impressed by, but I remember the poem the best; it was funny and it used the right math ideas about linear algebra to *be* funny, which essentially forces that the person *understood* the concepts. It was brilliant.

Please understand the point of this exercise, should you choose to do it: Pick any scientific discipline. I mean ANY. If it is scientific it's ok: how to build a bridge, how do design a new chemical, how to solve some hard algorithmic problem using computers (like how many stars the Netflix algorithm should predict for your enjoyment of the 1977 original version of Disney's Pete's Dragon, for example). Take 5 steps into your chosen scientific field and you will bump into linear algebra. That's the exercise. 1pt is essentially "write down in your own words what wiki has to say about it", 2pts is essentially "do something a little better but without any real math content", and then 3pts is "a pretty good job explaining how a specific concept from linear algebra is used in a scientific field you are interested in", where I will collect and grade these myself so it is up to my subjective expert opinion if what you say is a good job with the math explaining.

For 10/30 MATLAB #3 see other document.

For 11/6 **Part 0:** *Optional:* Create two **easy** or **medium difficulty** exam or quiz style problems using any of the concepts from the previous two weeks (check the schedule to determine the topics). Include a solution and follow the same instructions from last week.

Part 1: *This exercise will ask you to explore the equality* $Null(A^T) = (Col(A))^{\perp}$. Pick a vector u in $\mathbb{R}^4$ and a vector v such that u and v are orthogonal; then find a vector w which is orthogonal to both u and v. Finally, find a vector z which is orthogonal to u, v, w. For the last two steps, you should realize that you are solving a system of linear equations, and if you write them down that the system is represented by A^T where the columns of A are the vectors you are trying to be orthogonal to. Now, verify with computations that the set $\{u, v, w, z\}$ is linearly independent. If any of the vectors u, v, w, z are **scalars of the standard basis vectors** e_1, e_2, e_3, e_4 then start over. Set the matrix $P = [u\ v\ w\ z]$ and compute *without calculations* the vectors $P^{-1}u, P^{-1}v, P^{-1}w$, and $P^{-1}z$.

For 11/13 **Part 0:** *Optional:* Create two **easy** or **medium difficulty** exam or quiz style problems using any of the concepts from the previous two weeks (check the schedule to determine the topics). Include a solution and follow the same instructions from last week.

Part 1: Pick any three vectors u, v, w in $\mathbb{R}^4$ which are *linearly independent* but **not orthogonal** and a vector b which is **not** in the span of u, v, w. If any of your vectors u, v, w

are scalars of the standard basis vectors e_1, e_2, e_3, e_4 then start over. Let $W = \text{span}\{u, v, w\}$. Compute the orthogonal projection $\hat{b}$ of b onto the subspace W in two ways: (1) using the basis $\{u, v, w\}$ for W, and (2) using an orthogonal basis $\{u', v', w'\}$ obtained from $\{u, v, w\}$ via the Gram-Schmidt process. Finally, explain in a few words why the two answers differ, and explain why only ONE answer is correct.

For 11/20: **Part 0:** Create two **easy** or **medium difficulty** exam or quiz style problems using any of the concepts from the previous two weeks (check the schedule to determine the topics). Include a solution and follow the same instructions from last week.

Part 1: Create two **easy** or **medium difficulty** exam or quiz style problems connecting any TWO concepts from DIFFERENT MIDTERMS (check the schedule to determine the topics). Include a solution and follow the same instructions from last week.

EXTRA: **Part 0:** Create FOUR **easy** or **medium difficulty** exam or quiz style problems connecting any TWO concepts from DIFFERENT MIDTERMS (check the schedule to determine the topics). Include a solution and follow the same instructions from last week.