Handwritten Homework Assignments - Exploration for MATH 1554

For each assignment, complete the questions on a separate sheet of paper and put your name on it. Write neatly and use complete sentences where necessary. You must submit original work, but I'm okay with you all working together to share ideas. Handwritten homework assignments are due on Fridays. Please turn in handwritten homework in Gradescope (access the first time through Canvas) every Friday by midnight.

Week 1: *Practice sketching in 3D.* In $\mathbb{R}^3$ (so using three coordinate axes) sketch the following, making a new sketch for each part (i)-(v):

(i) the plane $z = 0$,

(ii) the plane $z = 2$,

(iii) the plane $y = -3$,

(iv) the plane $x + y + z = 0$,

(v) the intersection of the planes $z = 2$ and $x = 0$.

In each case, you are practicing drawing an accurate, representative graph of the plane of points which satisfy the given equation in the variables x, y, and z.

Sal says: For instance, for part (iii) several points on the plane are $(0, -3, 0), (1, -3, 2), (-1, -3, 0)$, etc., and the collection of *all* points which satisfy the equation is the thing I want you to draw. I understand that I never showed you how to do this, but if you just think about what you understand to be the meaning of a statement like "the line $y = 3x + 1$ in the xy-plane" it should be somewhat obvious what I'm asking you to do, even if its not clear *how* to accomplish the task. I think just trying something (really actually trying) is actually what I want you to do (rather than see in 5 seconds how *I* would do it and then sort of losing out on the whole point, which is that I want you to *think* about what these questions even *mean*!) Hope that is acceptable.

Week 2: *Practice with span, linear combination, and inconsistent systems.* (1) Choose two vectors v_1, v_2 in $\mathbb{R}^2$ and another vector b also in $\mathbb{R}^2$. Find scalars x, y in $\mathbb{R}$ such that $xv_1 + yv_2 = b$ (if this is not possible, pick other v_1, v_2, b vectors). Illustrate the vector equation you just solved by graphing the vectors v_1, v_2, b in the $x - y$-plane, and be sure to illustrate how b is obtained by adding a scalar of one vector to the other. (2) Repeat part 1 with vectors v_1, v_2, b in $\mathbb{R}^3$ that give a consistent system. (3) Why is part 2 more difficult than part 1? Explain clearly using complete sentences.

NOTE: For full credit, your answer must satisfy the following constraints: v_1, v_2 are not allowed to be scalar multiples of $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$; neither of the values of x, y is allowed to be $-1, 0$, or 1 for $\mathbb{R}^2$, and you can have at most one 0, and one 1 of these for R^3. The vectors v_1, v_2 should be "relatively random looking", subject to my discretion. Don't overthink part (3): a single sentence will suffice.

Week 3: *Learn some basics of MATLAB.* See Announcement in Canvas for MATLAB #1 Exploration.

Week 4: *Practice with transformations.* For each matrix A below, (0) state the domain and codomain of T_A, (1) find $T_A(e_1), T_A(e_2)$, (2) find $T_A(v), T_A(w)$, (3) describe in a few words what the transformation is doing, and (4) determine if the associated transformation is one-to-one/onto or not, and (5) give the matrix an appropriate "name" (fine to be silly name like, e.g., "the x-zero-er" for projection to y-axis"). For the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

1. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

9. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, for the problems below use

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 1 \\ \text{-3} \end{bmatrix}$$

1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sal says: For the "name your matrix" this is a bit silly, and that's ok. Just come up with a creative name that makes sense to you, and don't worry about it too much!

Week 5: *Practice matrix algebra "fake truths".* For full credit, correctly indicate which problem you are solving by writing the statement you are answering (like "$AB = 0$ and $A \neq 0$,$B \neq 0$"). For grading purposes, please try to write the problems in the same order as listed here. The matrix 0 is the zero matrix and the matrix I is the identity matrix.

For each problem find square matrices which satisfy the given conditions. You don't have to justify *how* you found the matrices for each problem, but you *must verify the equality with calculations* in each case. Just show the matrices A, B, C and the given products.

The following restrictions are required for each problem:
No matrix A, B, or C can be diagonal, none can be equal or a scalar multiple of each other, and no product can be the zero matrix (except *(d)*) or scalar multiple of the identity matrix (except *(e)*). All of the below are possible with these restrictions.

(a) $AB = BA$ but neither A nor B is 0 nor I, and $A \neq B$.

(b) $AB \neq BA$.

(c) $AB = AC$ but $B \neq C$, and the matrix A has *no zeros entries*.

(d) $AB = 0$ but neither A nor B is 0.

(e) $AB = I$ but neither A nor B is I.

Week 6: MATLAB #2. See Announcement in Canvas.

Week 7: For the following exploration, your matrix A needs to be an invertible 3×3 matrix and relatively random looking. So, in particular it should have:

- not too many 1's or 0's,
- not be in REF or RREF,
- not be a scalar multiple of the identity matrix, or have too many columns which are scalar multiples of e_i the standard basis vectors, and additionally
- should not have a very ugly answer to *(Part i)*.

Part (i) is not related to Parts (ii) and (iii), but Parts (ii) and (iii) are related to each other.

Part i) Write down a system of three equations in three variables, whose augmented matrix $[A|b]$ would then be 3×4. Find the determinant of the matrix A whose entries are the coefficients of the system. Ensure the determinant is non-zero and find the inverse of the matrix and also calculate $A^{-1}b$ *(you may use matlab or an online calculator for this part)*. Describe how $A^{-1}b$ relates to the system $[A|b]$ in a few words.

Part ii) Using either the matrix A you wrote down in *part (i)* or a new matrix A which also satisfies the bullet points of being relatively random looking, row reduce the matrix A to rref. What is the rref of A? Following the row operations you made to reduce A to rref, state the determinant of each elementary matrix. That is, if I is the rref of A and $A \sim E_1A \sim E_2E_1A \sim \cdots \sim E_n \cdots E_2E_1A = I$ are the matrices you got when you row reduced A to I, then calculate or otherwise find the determinant $\det A$, $\det E_1$, $\det E_2$, $\ldots$, $\det E_n$, and $\det I$.

Part iii) Compute the determinant of each of the intermediate matrices E_1A, E_2E_1A, $\ldots$, $E_n \cdots E_2E_1A$. Compare your result to what you did in *part (ii)*. Try to state the relationship between A and the intermediate row equivalent matrices, in terms of the determinants of the E_i's. Formulate a relationship between det(A) and the product of the $\det(E_i)$'s.

Week 8: Two separate parts.

1. Find a 2×2 matrix A *with real entries* with **no real eigenvalues** and show that it is correct by finding the characteristic polynomial and explaining why it has no real roots.

2. Find all eigenvalues and a corresponding eigenvector for each matrix below **without calculations** by thinking it out using the linear transformation's **geometric interpretation**. Write a few words (like 5) in each case explaining why your eigenvector/eigenvalue pair works. For each matrix, graph each eigenvector and it's image after the transformation as well as a random NON-eigenvector. Check your graph is accurate by showing the matrix multiplication.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A = id$.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T_A =$projection onto x-axis.

(c) $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, $T_A =$rotation by θ.
(and state the values of θ for which A has eigenvectors)

(d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T_A =$reflect about the line "$y = x$".

(e) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A =$stretch in x-direction.

(f) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $T_A =$shear.

(g) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $T_A =$stretch z-axis.

(h) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $T_A =$rotate by $90°$ about the x-axis.

Week 9: **Part 1:** Show that there is no relationship between any kind of row operation and the eigenvalues of the matrices involved as follows. For each of the **three types** of row operation $cR_i + R_j \to R_j$, $cR_j \to R_j$, and $R_i \leftrightarrow R_j$ which are *adding two rows*, *multiplying a row by a scalar*, and *switching two rows*: Find four matrices (for a **total of 12 matrices**) A, B, C, D such that $A \neq B$ and $C \neq D$ and also $A \sim B$ and $C \sim D$, the matrix A is row equivalent to B via exactly **one** row operation of the appropriate type, and also C is row equivalent to D via exactly **one** row operation of the appropriate type, and such that A and B have the exact same eigenvalues but C and D have different eigenvalues, and **be sure to state all eigenvalues**. Conclude that there is no relationship whatsoever between two matrices "being row equivalent" and "have the same eigenvalues".

Part 2: Let λ be a real eigenvalue of a matrix with real entries A. Show that the set $V_\lambda = \{x : Ax = \lambda x\}$ is a subspace of $\mathbb{R}^n$. If you reduce your solution to a question about null spaces, be sure to include prove that null spaces are subspaces (but that's fine if you want to do it that way so long as your argument is clear, and correct of course).
Hint: check the definition of subspace.

Part 3: Why is the set of eigenvectors of A corresponding to eigenvalue λ NOT a subspace? Why does this not contradict what you were asked to do in Part 2?
Hint: check the definition of eigenvector.

Week 10: **Part 1:** NOTE: This week's assignment doesn't have to be handwritten. (For 1pt) Write a paragraph which explains how eigenvectors/eigenvalues or some other topic from the course are used in a field which interests you. Be specific and put some thought into this. (For 2pt) If your paragraph is not essentially the thing on wikipedia about how bridges have something to do with eigenvalues, but actually give some details or original content then you get 2pts instead of 1pt. (For 3pt) If you also support your research with real math or alternately something creative. This has to include *some mathematical content* but can take *any form whatsoever*. Last year's submissions included a poem, several posters, a few slide-show presentations like using PowerPoint, etc., and quitte a few of them actually were pretty decent research project results that I was quite impressed by, but I remember the poem the best; it was funny and it used the right math ideas about linear algebra to *be* funny, which essentially forces that the person *understood* the concepts. It was brilliant.

Please understand the point of this exercise, should you choose to do it: Pick any scientific discipline. I mean ANY. If it is scientific it's ok: how to build a bridge, how do design a new chemical, how to solve some hard algorithmic problem using computers (like how many stars the Netflix algorithm should predict for your enjoyment of the 1977 original version of Disney's Pete's Dragon, for example). Take 5 steps into your chosen scientific field and you will bump into linear algebra. That's the exercise. 1pt is essentially "write down in your own words what wiki has to say about it", 2pts is essentially "do something a little better but without any real math content", and then 3pts is "a pretty good job explaining how a specific concept from linear algebra is used in a scientific field you are interested in", where I will collect and grade these myself so it is up to my subjective expert opinion if what you say is a good job with the math explaining.

Week 11: MATLAB #3 see other document.

Week 12: **Part 1:** *This exercise will ask you to explore the equality* $Null(A^T) = (Col(A))^{\perp}$. Pick a vector u in $\mathbb{R}^4$ and a vector v such that u and v are orthogonal; then find a vector w which is orthogonal to both u and v. Finally, find a vector z which is orthogonal to u, v, w. For the last two steps, you should realize that you are solving a system of linear equations, and if you write them down that the system is represented by A^T where the columns of A are the vectors you are trying to be orthogonal to. Now, verify with computations that the set $\{u, v, w, z\}$ is linearly independent. If any of the vectors u, v, w, z are **scalars of the standard basis vectors** e_1, e_2, e_3, e_4 then start over. Set the matrix $P = [u\ v\ w\ z]$ and compute *without calculations* the vectors $P^{-1}u, P^{-1}v, P^{-1}w$, and $P^{-1}z$.

Week 13: **Part 1:** Pick any three vectors u, v, w in $\mathbb{R}^4$ which are *linearly independent* but **not orthogonal** and a vector b which is **not** in the span of u, v, w. If any of your vectors u, v, w are scalars of the standard basis vectors e_1, e_2, e_3, e_4 then start over. Let $W = \text{span}\{u, v, w\}$. Compute the orthogonal projection $\hat{b}$ of b onto the subspace W in two ways: (1) using the basis $\{u, v, w\}$ for W, and (2) using an orthogonal basis $\{u', v', w'\}$ obtained from $\{u, v, w\}$ via the Gram-Schmidt process. Finally, explain in a few words why the two answers differ, and explain why only ONE answer is correct.

Week 14: NOTE: Please do not use any calculators, notes, textbook, or outside help for these problems. You will get FULL CREDIT so long as you attempt both problems and show all of your steps. If you have no idea how to solve them, please do the best you can to indicate any of the terms or definitions that you do understand what they mean. Please treat these as exam problems, as we have been asked to assess the 1554 students ability to solve these two problems this semester.

PLEASE ASSIGN PAGES IN GRADESCOPE TO EACH PROBLEM TO ASSIST WITH GRADING. Thank you for your help.

Problem 1: Find all eigenvalues, and an associated eigenvector for each eigenvalue, of the matrix A below.

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Problem 2: Let H be the subspace shown below. Find an orthonormal basis of the subspace.

$$H = \text{span}\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \right\}$$

Note: Please leave your answer with radicals. The numbers are actually not terrible except for the radical.